Introduction to Tensors and Exterior forms

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1 Linear Functionals and Dual Space

Let E be a vector space with standard basis $\{e_1, \dots e_n\}$, then a generic vector in E has the unique expansion $v = v \sum_j e_j v^i = \sum_j e^j v_j$.
t A real linear functional α on E is a real-valued linear function α that is a linear transformation $\alpha : E \to \mathbb{R}$. So we have

$$\alpha(a\mathbf{v} + b\mathbf{w}) = a\alpha(\mathbf{v}) + b\alpha(\mathbf{w}) \tag{1}$$

for real numbers a, b and vector $\mathbf{v}, \mathbf{w}$. By induction we have for any basis $\mathbf{e}$

$$\alpha(\sum e_j v^j) = \sum \alpha(e_j)v^j \tag{2}$$

so what we have is

$$\alpha(\mathbf{v}) = \sum a_j v^j \qquad \text{where } a_j := \alpha(e_j) \tag{3}$$

Definition The collection of all linear functionals α on a vector space E form a new vector space E^*, the **dual space** to E. under the operations

$$\begin{aligned}(\alpha + \beta)(v) :=& \alpha(v) + \beta(w) \qquad \alpha, \beta \in E^*, \qquad v \in E \\ (c\alpha)(v) :=& c\alpha(v) \qquad c \in \mathbb{R}\end{aligned}$$

we may define a **dual basis** $\{\sigma^1, \dots \sigma^n\}$ of E^* by putting

$$\sigma^i(e_j) = \delta^i_j \tag{4}$$

and so by linearity we have

$$\sigma^i(\sum e_j v^j) = \sum \sigma^i(e_j)v^j = v^i \tag{5}$$

this means we can re-write 1.2 as

$$\sum_j \alpha(e_j)\sigma^j(v) = \left(\sum_j \alpha(e_j)\sigma^j\right)(v) \tag{6}$$

If one looks at 1.2 and 1.3 it is easy to see that α and $\sum_j \alpha(e_j)\sigma^j$ are the same so

$$\alpha = \sum \alpha(e_j)\sigma^j \tag{7}$$

2 The Differential of a Function

Definition Let f: $M^n \to \mathbb{R}$. the **differential** of f at p (on a manifold), written as df, is the linear functional $df : M^n_p \to \mathbb{R}$ defined by

$$df(\mathbf{v}) = \mathbf{v}_p(f) \tag{8}$$

In the above defintion $\mathbf{v}$ is a vector at p which is a differential operator of functions near p. The above definition basis independent but if we choose local co-ordinates (1.8) becomes

$$df\left(\sum v^j \frac{\partial}{\partial x_j}\right) = \sum v^j \frac{\partial f}{\partial x_j} \tag{9}$$

in the above equation we are using $e_j = \frac{\partial}{\partial x_j}$ as a basis. If we consider the differential of a particular coordinate X^i we have

$$dx^i\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial x^i}{\partial x^j} = \delta^i_j \tag{10}$$

and in particular we have that

$$dx^i\left(\sum v^j \frac{\partial}{\partial x_j}\right) = \sum v^j \frac{\partial x^i}{\partial x_j} = v^i \tag{11}$$

but the above behavior was exactly we observed for the a basis in the dual space in (1.5) so we make the conclusion that $\sigma^i = dx^i$. The most general differential is then written as

$$\alpha = \sum_j \alpha(\frac{\partial}{\partial x_j})dx^i = \sum_j a_j dx^j \tag{12}$$

The linear functional α is called a **covariant** vector or **co-vector** and $\sum_j a_j(x)dx^j$ is a co-vector field.

Let U and V be different charts on a manifold then under a change of coordinates we have

$$dx^i_V = \sum_j \frac{\partial x^i_V}{\partial x^j_U} dx^j_U \tag{13}$$

but a general co-vector is $\sum a^V_i dx^i_V$ and using above equation we have $\sum a^V_i dx^i_V = \sum a^V_i \frac{\partial x^i_V}{\partial x^j_U} dx^j_U = \sum a^U_i dx^i_U$ so conclude that

$$a^U_j = \sum_i a^V_i \frac{\partial x^i_V}{\partial x^{j_U}} \tag{14}$$

compare this with a contravariant vector field which transforms as

$$X^i_V = \sum \frac{\partial x^i_V}{\partial x^j_U}(p_0) X^j_U \tag{15}$$

As a consequence of this formalism we can make sense of the gradient vector field

2.0.1 Gradient vector

Let M^n be a pseudo-Riemannian manifold and f be a differentiable function. The grad f= ∇f is the contravariant vector associated to the covector with df i.e $df(w) =< \nabla f, w >$

There is a correspondence between V and V^* and df is paired with ∇f in this correspondence. Consider the following:

$$< v, w >= \sum_{ij} v^i < e_i, e_j > w^j$$

Thinking about the $< v, w >$ keep v fixed and vary w. This way we can define a linear functional, $\nu(w) =< v, w >$. We expand the functional in some basis in the dual space $\nu = \sum_j v_j \sigma^i$ with $v_j = \nu(e_j) =< v, e_j >$. Now we have that $v^i = \sum g^{ij} v_j = \sum_j g^{ij} < v, e_j >$. So

$$\mathbf{v} = \sum_j v^i e_i = \sum_i (\sum_j g^{ij} < v, e_j >) e_i \tag{16}$$

Let us now investigate the definition of the gradient given the above discussion with (1.16) in mind

$$\nabla f = \sum_i (\nabla f)^i e_i = \sum_i (\sum_j g^{ij} < \nabla f, e_j >) e_i \tag{17}$$

and by definition $df(w) =< \nabla f, w >= w(f) = \sum_j w^j \frac{\partial f}{\partial x^j}$ so $< \nabla f, e_j >= e_j(f) = \frac{\partial f}{\partial x_j}$. We thus arrive the following equation for the gradient

$$(\nabla f)^i = \sum_j g^{ij} \frac{\partial f}{\partial x^j} \tag{18}$$

Note that in euclidean geometry df and ∇f have the same coordinates.

2.0.2 Pull Back of Covector

We can also talk about the differential of a covector. Suppose we have a map $\phi : M^n \to V^r$. Then the differential of the map ϕ denoted as ϕ_* takes one from the tangent space of M^n to the tangent space of V^r. The way define it is by choosing a curve $\gamma(t)$ such that $\gamma(0) = p_0$ and $\dot{\gamma}(0) = X$ where X is a vector field and calculating $(\phi \circ \gamma)' = \phi_* X = \frac{\phi(\gamma(t)}{dt}|_{t=0}$. The matrix components of this map are elements of the Jacobian i.e

$$(\phi_*)^\alpha_i = \frac{\partial \phi^\alpha}{\partial x^i}(p) \tag{19}$$

in terms of basis elements $\frac{\partial}{\partial x^i}$ at p and $\frac{\partial}{\partial y^i}$ at $\phi(p)$. So in terms of local coordinates we can write:

$$\phi_*(\frac{\partial}{\partial x^j}) = \sum_i \frac{\partial y^i}{\partial x^j}\frac{\partial}{\partial y^i} \tag{20}$$

Definition Let $\phi : M^n \to V^r$ be a smooth map of manifold and let $\phi(x) = y$. Let $\phi_* : M_x \to V_y$ be the differential of ϕ. The **pull-back** ϕ^* is the linear transformation taking *co vector* at y into *covector* at x, $\phi^* : V(y)^* \to M(x)^*$ defined by

$$\phi^*(\beta)(\mathbf{v}) := \beta(\phi_*(\mathbf{v}))$$

for all covectors β at y vectors $\mathbf{v}$ at x.

$$\begin{aligned}
\phi^*(\beta) &= \sum_j \phi^*(\beta)\left(\frac{\partial}{\partial x^j}\right) dx^j = \sum \beta\left(\phi_* \frac{\partial}{\partial x^j}\right) dx^j \\
&= \sum \beta \left(\sum_R \left(\frac{\partial y^R}{\partial x^j}\right) \frac{\partial}{\partial y^R}\right) dx^j \\
&= \sum_{jR} \frac{\partial y^R}{\partial x^j} \beta \left(\frac{\partial}{\partial y^R}\right) dx^j \\
&= \sum_{jR} b_R \left(\frac{\partial y^R}{\partial x^j}\right) dx^j \qquad \text{where } \beta = \sum_R b_R dy^R
\end{aligned}$$

3 Tensors

3.0.1 Covariant Tensors

Definition A **covariant tensor of rank r** is a **multilinear** real-valued function $Q : E \times E \times E \ldots \times E \to \mathbb{R}$

$$\begin{aligned}
Q(v_1, \ldots v_r) =& Q\left(\sum_{i_1} v^{i_1}\partial_i, \ldots, \sum_{i_r} v_r^{i_r}\partial_{i_r}\right) \\
&= \sum_{i_1 \ldots i_r} v_1^{i_1} \ldots v_r^{i_r} Q(\partial_{i_1}, \ldots \partial_{i_r}) \\
&= \sum_{i_1, \ldots i_r} Q_{i_r \ldots i_r} v_1^{i_1} \ldots v_r^{i_r}
\end{aligned}$$

where $Q_{i_r \ldots i_r} v_1^{i_1} \ldots v_r^{i_r} := v_1^{i_1} \ldots v_r^{i_r}$.

Collection of all rank r co-variant tensor form a vector space denoted as $E^* \otimes E^* \otimes \ldots \otimes E^* = \otimes^r E^*$

Let α and $\beta \in E^*$ we form a second rank tensor with tensor product . We need only to specify how $\alpha \otimes \beta : E \times E \to \mathbb{R}$. This we will be defined as

$$\alpha \otimes \beta(v, w) := \alpha(v)\beta(w) \tag{21}$$

3.0.2 Contravariant Tensor

Vector can be defined as a function acting on covectors

$$v(\alpha) := \alpha(v)$$

In component form $v(\alpha) = a_i v^i$.
Contravariant tensor is a multilinear function T on s-tuples of co-vectors:

$$T : E^* \times E^* \ldots E^* \to \mathbb{R}$$

$$T(\alpha_1, \ldots \alpha_s) = a_{1i_1} \ldots a_{si_s} T^{i_1 \ldots i_s} \tag{22}$$

where $T^{i_1 \ldots i_s} := T(dx^{i_1}, \ldots dx^{i_s})$

The space of contravariant tensors is $E \otimes E \otimes \ldots \otimes E = \otimes^r E$

3.0.3 Mixed Tensor

Definition A **mixed tensor** , r times covariant and s times contravariant, is a real multilinear function W

$$W : E^* \times E^* \times \ldots \times E^* \times E \times E \times \ldots \times E \to \mathbb{R}$$

on s-tuples of covectors and r-tuples of vectors

By multilinearity we have that

$$T(\alpha_1, \ldots \alpha_s, v_1, \ldots v_r) = a_{i_1} \ldots a_{i_s} W^{i_1 \ldots i_s}{}_{j_1 \ldots j_r} v_1^{j_1} \ldots v_r^{j_r} \tag{23}$$

where

$$W^{i_1 \ldots i_s}{}_{j_1 \ldots j_r} := W(dx^{i_1}, \ldots \partial_{j_r}) \tag{24}$$

3.0.4 Transformation Properties of Tensors

Under a change of bases, $\partial'_l = \partial_s (\frac{\partial x^s}{\partial x'^l})$ and $dx'^i = (\frac{\partial x'^i}{\partial x^c}) dx^c$

$$W'^{i \ldots j}{}_{k \ldots l} = W(dx'^i, \ldots dx'^j, \partial'_k, \ldots \partial'_l) \tag{25}$$

$$= \left(\frac{\partial x'^i}{\partial x^c}\right) \ldots \left(\frac{\partial x'^j}{\partial x^d}\right) \left(\frac{\partial x^r}{\partial x'^k}\right) \ldots \left(\frac{\partial x^s}{\partial x'^l}\right) W^{c \ldots d}{}_{r \ldots s} \tag{26}$$

Similar equations can be found for contravariant and contravariant tensors.

4 Grassmann (Exterior) Algebra

The grassmann or exterior algebra is a product that is a vast generalization of scalar and vector products in vector analysis. Before we discuss it, we first begin with a discussion of a simpler product, that is one defined with covectors.

Definition If $\alpha \in \bigotimes^p E^*$ and $\beta \in \bigotimes^q E^*$, then their **tensor product** $\alpha \otimes \beta$ is the covariant (p+q) -tensor defined by

$$\alpha \otimes \beta(\mathbf{v}_1, \dots \mathbf{v}_{p+q}) := \alpha(\mathbf{v_1}, \dots, \mathbf{v_p})\beta(\mathbf{v}_{p+1}, \dots \mathbf{v}_q)$$

Definition An **(exterior) p-form** is a covariant p-tensor $\alpha \in \bigotimes^p E^*$ that is anti-symmetric i.e

$$\alpha(\dots, \mathbf{v_r}, \dots, \mathbf{v_s}, \dots) = -\alpha(\dots, \mathbf{v_s}, \dots, \mathbf{v_r}, \dots)$$

in each pair of entries.

The collection of p-forms is a vector

$$\Lambda^p E^* = E^* \Lambda E^* \Lambda \dots \Lambda E^* \subset \bigotimes^p E^*$$

By defintion $\Lambda^1 E^* = E^*$the space of one forms and $\Lambda^0 E^* = \mathbb{R}$ 0-forms or scalars

4.0.1 Multi-index Notation

Since we are dealing with p-forms we need to simplify our notation. $I = (i_1, \dots i_p)$ where these are indices for a p-form. So for example let $\alpha \in \Lambda^p E^*$ and let $\partial_\mathbf{i}$ be a basis for E. Then α has n^p components denoted as

$$a_I = a_{i_1,\dots,i_p} = \alpha(\partial_{\mathbf{i_1}}, \dots \partial_{i_p}) = \alpha(\partial_I) \tag{27}$$

The indices in the above expressions and generally we be listed in strictly increasing order. If we are on an n-dimensional manifold, we ask how large the dimension of $\Lambda^p E^*$ is. where $p \leq n$. This amounts to a combinatorial problem whose answer is the binomial coefficient.

$$\dim \overset{p}{\Lambda} E^* = \frac{n!}{p!(n-p)!} \tag{28}$$

Since if an index repeats, the exterior form is zero, an exterior form where $p > n$ will be zero, since an index will have to repeat.

We wish to define a product for $\alpha \otimes \beta$ which is a (p+q) tensor. The problem is that this need not be skew symmetric in all indices, so it need not be a (p+q) form. This problem was solved by Grassmann who defined the following product, which we will call the **wedge product**

$$\alpha^1 \wedge \beta^1 := \alpha \otimes \beta - \beta \otimes \alpha \tag{29}$$

in particular we have that

$$\alpha^1 \wedge \beta^1(v,w) = \alpha(v)\beta(w) - \beta(v)\alpha(w) \tag{30}$$

So $\alpha \wedge \beta$ is not only a tensor but a 2 form. We now define a generalized kronecker delta

$$\begin{aligned} \delta^I{}_J := & 1 \text{ if J } = (j_1, \dots j_r) \text{ is an even permutation of } I = (i_1, \dots i_r) \\ = & -1 \text{ if Jvis an odd permutation of I} \\ = & 0 \text{ if J is not a permutation of I} \end{aligned}$$

We now define the permutation symbol

$$\epsilon_I = \epsilon_{i_1, \dots i_n} = \epsilon^I := \delta^I_{12\dots n} \tag{31}$$

which defines whether the n indices $i_1, \dots i_n$ form an even or odd permutation of $1, \dots n$. This appears in the definition of the determinant $detA = \epsilon_I A^{i_1}{}_1 A^{i_2}{}_2 \dots A^{i_n}{}_n$

We define the **exterior** or **wedge** or **Grassmann** product

$$\wedge : \overset{p}{\Lambda} E^* \times \overset{q}{\Lambda} E^* \rightarrow \overset{p+q}{\Lambda} E^*$$

Concretely this means:

$$\alpha \wedge \beta(\mathbf{v}_I) := \sum_K \sum_J \delta_I^{JK} \alpha(\mathbf{v}_J)\beta(\mathbf{v}_K) \tag{32}$$

where $I = (i_1, \dots i_{p+q}), J = (j_1, \dots j_p)$ and $K = (k_1, \dots k_q)$.
For example let dim E = 5 and if $e_1, \dots e_5$ is a basis for E then

$$\begin{aligned} (\alpha^2 \wedge \beta^1)_{523} = & \alpha^2 \wedge \beta^1(e_5, e_2, e_3) \\ = & \sum_{r<s} \sum_t \delta^{rst}_{523} \alpha_{rs} \beta_t \\ = & \delta^{253}_{523} \alpha_{25}\beta_3 + \delta^{352}_{523}\alpha_{35}\beta_2 + \delta^{235}_{523}\alpha_{23}\beta_5 \\ = & -\alpha_{25}\beta_3 + \alpha_{35}\beta_2 + \alpha_{23}\beta_5 \end{aligned}$$

One may consider the vector space of all forms over E^*.

$$\overset{*}{\Lambda} E^* := \left(\overset{0}{\Lambda} E^*\right) \oplus \left(\overset{1}{\Lambda} E^*\right) \oplus \dots \oplus \left(\overset{n}{\Lambda} E^*\right)$$

.

This is the Grassmann or exterior algebra over E^* with $dim \overset{*}{\Lambda} E^* = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$

To show associativity of the algebra we use the following result namely:

$$\sum_J = \delta^{IJ}_M \delta^{KL}_J = \delta^{IKL}_M \tag{33}$$

We now show associativity:

$$\begin{aligned}[\alpha^p \wedge (\beta^q \wedge \gamma^r)]_M &= \sum_{IJ} \delta_M^{IJ} \alpha_I (\beta^q \wedge \gamma^r)_J \\ &= \sum_{IJKL} \delta_M^{IJ} \alpha_I \delta_J^{KL} \beta_K \gamma_L \\ &= \sum_{IKL} \delta_M^{IKL} \alpha_I \beta_L \gamma_L\end{aligned}$$

$$\begin{aligned}[(\alpha^p \wedge \beta^q) \wedge \gamma^r]_M &= \sum_{NL} (\alpha^p \wedge \beta^q)_N \gamma_L \delta_M^{NL} \\ &= \sum_{NLIK} \delta_N^{IK} \alpha_I \beta_K \delta_M^{NL} \delta_L \\ &= \sum_{IKL} \sum_{M}^{IKL} \alpha_I \beta_K \gamma_L\end{aligned}$$

Suppose all the forms are 1-forms then

$$\begin{aligned}\alpha_1 \wedge \alpha \wedge \ldots \alpha_r(\mathbf{v}_1 \ldots \mathbf{v_r}) &= \sum_I \delta^I_{i_1,\ldots i_r} \alpha_1(\mathbf{v}_{i(1)}) \alpha_2(\mathbf{v}_{i(2)}) \ldots \alpha_r(\mathbf{v}_{i(r)}) \\ &= det[\alpha_j(\mathbf{v}_i)]\end{aligned}$$

and let $(\sigma^1, \ldots \sigma^n)$ be a basis of 1 forms dual to $(e_1, \ldots e_n)$ and let $\sigma^I \implies \sigma^{i_1} \wedge \ldots \wedge \sigma^{i_r}$ and $\sigma^I(e_J) = \delta^I{}_J$ then $\alpha^p = \sum_I a_I \sigma^I$. Now we consider an n-tuple of 1 forms $\tau^1, \tau^2 \ldots \tau^n$ and expand them in terms of the basis i.e $\tau^i = T^i{}_I \sigma^j$(we are not assuming any scalar product) then:

$$\begin{aligned}\tau^I &= \sum_J T^1_{j_1} T^2_{j_2} \ldots T^n_{j_n} \delta^J_I \sigma^I \\ &= det(T) \sigma^I \\ &= det(T) \sigma^1 \wedge \ldots \sigma^n\end{aligned}$$

5 Exterior Differentiation

The exterior derivative is a powerful form of differentiation of p-forms that in a sense generalizes the different kinds of differentiation one meets in three dimensional euclidean space. We now present a theorem that we shall not prove that introduces the **exterior derivative**.

Theorem 5.1 *There is a unique operator, exterior differentiation,*

$$d : \overset{p}{\Lambda} M^n \rightarrow \overset{p+1}{\Lambda} M^n$$

satisfying
i) d is additive, $d(\alpha+\beta) = d\alpha + d\alpha\beta$
ii) d α^0 *is the usual differential of the function* α^0
iii) $d(\alpha^p \wedge \beta^q) = d\alpha^p \wedge \beta^q + (-1)^p \alpha^p \wedge d\beta^q b$
iv) $d^2\alpha := d(d\alpha) = 0$*, for all forms* α

We instead introduce the operator by carrying specific computations in $\mathbb{R}^3$

5.0.1 Examples in Three Dimensions

Let $\mathbf{x} = x, y, z$ be any (perhaps curvilinear) coordinate system in $\mathbb{R}^3$. The differential of a function $f = f^0$ is

$$df^0 = \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z}\right) dz \tag{34}$$

If the coordinates are cartesian then we have that $df = \nabla f . d\mathbf{x}$. Considering a 1-form in general form i.e $\alpha^1 = a_1(\mathbf{x})dx + a_2(\mathbf{x})dy + a_3(\mathbf{x})dz$ then

$$\begin{aligned} d\alpha^1 &= da_1 \wedge dx + da_2 \wedge dy + da_3 \wedge dz \\ &= [\left(\frac{\partial a_1}{\partial x}\right) dx + \left(\frac{\partial a_1}{\partial y}\right) dy + \left(\frac{\partial a_1}{\partial z}\right) dz] \wedge dx \\ &+[\left(\frac{\partial a_2}{\partial x}\right) dx + \left(\frac{\partial a_2}{\partial y}\right) dy + \left(\frac{\partial a_2}{\partial z}\right) dz] \wedge dy \\ &+[\left(\frac{\partial a_3}{\partial x}\right) dx + \left(\frac{\partial a_3}{\partial y}\right) dy + \left(\frac{\partial a_3}{\partial z}\right) dz] \wedge dz \\ &= (\partial_y a_3 - \partial_z a_2) dy \wedge dz + (\partial_z a_1 - \partial_x a_3) dz \wedge dx + (\partial_x a_2 - \partial_y a_1) dx \wedge dy \end{aligned}$$

So in cartesian coordinates we have that

$$d(\mathbf{A} \cdot d\mathbf{x}) = (curl\mathbf{A}) \cdot d\mathbf{S} \tag{35}$$

for a 2-form $\beta^2 = b_1(dx \wedge dy) + b_2(dx \wedge dz) + b_3(dy \wedge dz) = \mathbf{B} \cdot d\mathbf{S}$ then

$$\begin{aligned} \beta^2 &= db_1 \wedge dx \wedge dy + db_2 \wedge dx \wedge dz + db_3 \wedge dy \wedge dz \\ &= \frac{\partial b_1}{\partial z} dz \wedge dx \wedge \partial dy + \frac{\partial b_2}{\partial y} dy \wedge dx \wedge \partial dz + \frac{\partial b_3}{\partial x} dx \wedge dy \wedge \partial dz \\ &= (\nabla \cdot \mathbf{B}) d\mathbf{V} \end{aligned}$$

NOTE: We have already laid the ground for generalizing and combining the divergence theorem and stokes theorem learned in multivariable calculus. These are the identities that appear as the integrands.

6 Interior Product and Vector Analysis

Another operation we can talk about for p-forms is a generalized notion of contracting a tensor. This notion is encapsulated in the definition of an interior product, which will be presented as a theorem and will not be proved.

Definition If $\mathbf{v}$ is a vector and α is a p-form, their **interior product** (p-1) form $i_{\mathbf{v}}\alpha$ is defined by

$$
\begin{aligned}
i_{\mathbf{v}}\alpha^0 &= 0 \\
i_{\mathbf{v}}\alpha^1 &= \alpha(\mathbf{v}) \\
i_{\mathbf{v}}\alpha^p(\mathbf{w_2}, \dots \mathbf{w_p})b &= \alpha^p(\mathbf{v}, \mathbf{w_2}, \dots \mathbf{w_p})
\end{aligned}
$$

$i_{\mathbf{A+B}} = i_{\mathbf{A}} + i_{\mathbf{B}}$ and $i_{\mathbf{aA}} = a i_{\mathbf{A}}$. Sometimes this the interior product will be referred to as $i(\mathbf{v})$

We present the following theorem which will not be proved.

Theorem 6.1 *$i_{\mathbf{v}}$: $\Lambda^p \to \Lambda^{p-1}$ is an **antiderivation** i.e ,*

$$i_{\mathbf{v}}(\alpha^p \wedge \beta^q) = [i_{\mathbf{v}}\alpha^p] \wedge \beta^q + (-1)^p \alpha^p \wedge [i_{\mathbf{v}}\beta^q]$$

Again, we give an introduction to the product by giving specific computations.
Let $E = \mathbb{R}^3$ with the basis e_1, e_2, e_3 and cobasis being e^1, e^2, e^3. Suppose $\alpha \in \Lambda^2(\mathbb{R}^3)$ more specifically $\alpha = e^3 \wedge e^2$ and $\mathbf{v} = e_1$ and $w \in \mathbb{R}^3$ Then computing the interior product $i_{\mathbf{v}}\alpha$ goes as follows:

$$
\begin{aligned}
i_{\mathbf{v}}\alpha(w) =& \alpha(\mathbf{v}, w) \\
=& e^3 \wedge e^2(e_1, w) \\
=& e^3(e_1)e^2(w) - e^2(e_1)e^3(w) \\
=& 0.
\end{aligned}
$$

if we change v so that $v = e_2$, the computation proceeds as follows

$$
\begin{aligned}
i_{\mathbf{v}}\alpha(w) =& \alpha(\mathbf{v}, w) \\
=& e^3 \wedge e^2(e_2, w) \\
=& e^3(e_2)e^2(w) - e^2(e_2)e^3(w) \\
=& -e^3(w)
\end{aligned}
$$

6.0.1 Reformulating Vector Analysis

The machinery for dealing with differential forms (what we have been calling p-forms) offers and very powerful way of dealing with vectors in three dimensions and makes otherwise tedious calculations "trivial". If this is to be done we need a way of translating operations for p-forms in terms of operations one uses in three dimensions.

What is the 1-form that corresponds to vectors ?

Roughly speaking, to every vector in $\mathbb{R}^3$ we associate a certain 1-form. More specifically, in section 1.2 we introduced the dual vector $\nu =<, v>$ since $\nu(w) =< w, v >$ so the correspondence is

$$\mathbf{v} \Leftrightarrow v_1 dx^1 + v_2 dx^2 + v_3 dx^3$$

What is the 2-form we associate with vectors?

Looking at the expressions derived for the divergence of vectors we see that a **volume form** for $\mathbb{R}^3$ is a associated with a 3-form. But we need a two form, so it turns out that we can use the interior product to reduce the three form to a 2-form. One might ask why we do not straight away simply use a 2-form. This is because the 2 form we want should care about the orientation of our space. So it turns out what we really need is a pseudo 2-form $\nu^2 := i_{\mathbf{v}} vol^3$ where vol^3 is a volume form. We justify the statement by carrying out the following computation

$$\begin{aligned}
i_{\mathbf{v}} \sqrt{g(u)} du^1 \wedge du^2 \wedge du^3 =& \sqrt{g} \sum v^i i(\partial_i)(du^1 \wedge du^2 \wedge du^3) \\
i(\partial_i)(du^1 \wedge du^2 \wedge du^3) =& du^1(\partial_i) du^2 \wedge du^3 - du^2(\partial_i) du^1 \wedge du^3 + du^1 \wedge du^2 du^3(\partial_i) \\
=& \delta^1{}_i du^2 \wedge du^3 - \delta^2{}_i du^1 \wedge du^3 + \delta^3{}_i du^1 \wedge du^2
\end{aligned}$$

So to the vector $\mathbf{v}$ we associate the pseudo 2-form

$$\mathbf{v} \Leftrightarrow \nu^2 := i_{\mathbf{v}} vol^3$$

where

$$i_{\mathbf{v}} vol^3 = \sqrt{g}(v^1 du^2 \wedge du^3 + v^2 du^1 \wedge du^3 + v^3 du^1 \wedge du^2) \tag{36}$$

In $\mathbb{R}^3$ given two vectors $\mathbf{v}$ and $\mathbf{w}$ with associated covectors $\nu^1 =<, \mathbf{v} >$, $\omega^1 =<, \mathbf{w} >$ we know that

$$< \mathbf{v}, \mathbf{w} >= i_{\mathbf{v}} \omega^1 \tag{37}$$

We can also associate with them their 2-forms ν^2 and ν^2 and we have that

$$\nu^1 \wedge \omega^2 =< \mathbf{v}, \mathbf{w} > vol^3 \tag{38}$$

The proof of the above expression is

$$\begin{aligned}
\nu^1 \wedge \omega^2 =& \nu^1 \wedge i_{\mathbf{w}} vol^3 \\
=& i_{\mathbf{w}} vol^3 \wedge \nu^1 \\
=& i_{\mathbf{w}}(vol^3 \wedge \nu^1) + vol^3 \wedge i_{\mathbf{w}} \nu^1 \\
=& i_{\mathbf{w}} vol^3
\end{aligned}$$

One operation in $\mathbb{R}^3$ is the cross-product. Calculating the cross-product of two vectors we know that the components are the same as those of $\nu^1 \wedge \omega^2$. So one would like to say that we associate $\mathbf{v} \times \mathbf{w}$ to the 2-form $\nu^1 \wedge \omega^2$. But we only have a pseudo 2-form so instead we say that we associate the pseudovector $\mathbf{v} \times \mathbf{w}$ with the 2-form $\nu^1 \wedge \omega^2$.

$$i_{\mathbf{v}\times\mathbf{w}} = \nu^1 \wedge \omega^1 \tag{39}$$

Not often taught is that the cross-product is defined as the unique vector such that

$$< \mathbf{v} \times \mathbf{w}, \mathbf{c} >= vol^3(\mathbf{v}, \mathbf{w}, \mathbf{c}) \tag{40}$$

We also look for the 1-form associated with the cross-product of $\mathbf{v} \times \mathbf{w}$. We start with (1.40)

$$\begin{aligned} < \mathbf{v} \times \mathbf{w}, \mathbf{c} >= vol^3(\mathbf{v}, \mathbf{w}, \mathbf{c}) &= -vol^3(\mathbf{w}, \mathbf{v}, \mathbf{c}) \\ &= -[i_{\mathbf{w}}(vol^3)](\mathbf{v}, \mathbf{c}) \\ &= -\omega^2(\mathbf{v}, \mathbf{c}) \\ &= -i_{\mathbf{v}}\omega^2(\mathbf{c}) \end{aligned}$$

Thus we have that

$$i_{\mathbf{v}}\omega^2 \text{ is the covariant version of } \mathbf{v} \times \mathbf{w} \tag{41}$$

We can now apply the above formalism to do non-trivial vector operations
Example Calculations

1.Calculate $A \times (B \times C)$
A, B, C go into their corresponding 1-forms i.e B$\Leftrightarrow$ β^1, C $\Leftrightarrow$ γ^1 so that the expression becomes $i_A(\beta^1 \wedge \gamma^1) = i_A(\beta^1)\gamma^1 - \beta^1 i_A(\gamma^1) = (A \cdot B)C - B(A \cdot C)$
.
2. Show $A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$
We associate with $C \Leftarrow \gamma^2, B \Leftrightarrow \beta^2, A \Leftrightarrow \alpha^2$
$A \cdot (B \times C) \Leftrightarrow -i_A(i_B\gamma^2) = -i_B i_A(\gamma^2) = -B \cdot (A \times C) = B \cdot (C \times A)$
$B \cdot (C \times A) \Leftrightarrow i_B i_C \alpha^2 = -i_C i_B(\alpha^2) = -C \cdot (B \times A)$.
This shows we have vector algebra neatly in our hands.

What about vector calculus?
We defined $df =<, \nabla f >$, now we define the curl $\mathbf{A}$ by using $\mathbf{A} \Leftrightarrow \alpha^1$ and then curl $\mathbf{A} \Leftrightarrow d\alpha^1$. These identifications are inspired by the calculations we did with the exterior differential operator. Thus

$$d\alpha^1 = i_{curl\mathbf{A}} vol^3 \tag{42}$$

and we define div $\mathbf{B}$ by using $\mathbf{B} \Leftarrow \beta^2$ and then

$$d\beta^2 = (div\mathbf{B})vol^3 \tag{43}$$

We can now write down general expressions for the divergence of a vector field without reference to any specific coordinate system

$$\begin{aligned}
d(i_B vol^3) =& d[\sqrt{g}b^1 du^2 \wedge du^3 + \sqrt{g}b^2 du^3 \wedge du^1 + \sqrt{g}b^3 du^1 \wedge du^2] \\
=& [\frac{\partial}{\partial u^1}(\sqrt{g}b^1) + \frac{\partial}{\partial u^2}(\sqrt{g}b^2) + \frac{\partial}{\partial u^3}(\sqrt{g}b^3)] du^1 \wedge du^2 \wedge du^3 \\
=& \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i}(\sqrt{g}b^i)\sqrt{g} du^1 \wedge du^2 \wedge du^3
\end{aligned}$$

So

$$div\mathbf{B} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i}(\sqrt{g}b^i) \tag{44}$$

For a scalar function f we associate with it a pseudo 3 form $fvol^3$.
One can use the above expression for the divergence and the expression for the gradient found in (1.2) to write down a general expression for the laplacian.

$$\nabla^2 f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i}(\sqrt{g}g^{ij}\frac{\partial f}{\partial x^i}) \tag{45}$$

3. Calculate $\nabla \cdot (A \times B)$
$div(A \times B)vol^3 = d(\alpha^1 \wedge \beta^1) = d\alpha^1 \wedge \beta^1 - \alpha^1 \wedge d\beta^1$ using (1.38) we have $< curlA, B > vol^3 - < A, curlB > vol^3$. Thus we have that $\nabla \cdot (A \times B) = (\nabla \times A) \cdot B - A \cdot (\nabla \times B)$

4. Calculate $\nabla(fg)$
$\nabla(fg) \Leftrightarrow d(fg) = dfg + fdg \Leftrightarrow \nabla fg + f\nabla g$

5. Calculate $\nabla \cdot (f\mathbf{B})$
$div(f\mathbf{B})vol^3 = d(f\beta^2) = df \wedge \beta^2 + fd\beta^2 =< \mathbf{B}, \nabla f > +f \ \ div\mathbf{B} \ \ vol^3$ So,
$\nabla \cdot (f\mathbf{B}) = \nabla f \cdot \mathbf{B} + f\nabla \cdot \mathbf{B}$

6. Calculate $\nabla \times (f\mathbf{A})$
$\nabla \times \Leftrightarrow d$ so $\nabla \times (f\mathbf{A}) \Leftrightarrow d(f\alpha^1) = df \wedge \alpha^1 + fd\alpha^1 \Leftrightarrow \nabla f \cdot \mathbf{A} + f\nabla \times \mathbf{A}$