On the way to study Lie groups and their representations we make a short detour into the study of lie algebras.

1 Introduction

Let F be a field. A *Lie Algebra* over F is a a F-vector space K, together with a bilinear map, the *lie bracket* :

$$L \times L \to L \qquad (x, y) \longmapsto [x, y] \tag{1}$$

satisfying the following properties

- .[x,x] =0 for all $x \in L$
- .[x,[y,z]] + [y,[z,x]] + [z,[x,y]]=0 for all $x, y, z \in L$

Some examples of lie algebras

1. Let F = $\mathbb{R}$. The vector product (x,y) $\longmapsto x \wedge y$
2. Any vector space V has a lie bracket defined by [x,y]=0 for all $x, y \in V$
3. Set of all linear maps from V $\to$ V. This is a vector space over F known as the general linear algebra [x,y] $\equiv x \circ y - y \circ x$ for all x, y $\in gl(V)$ where $\circ$ denote the composition of maps

2 Subalgebras and Ideals

Lie Subalgebra of L is a vector space $K \subseteq L$ such that [x,y] $\in K$ for all x,y $\in K$.

Ideal of a lie algebra L is a subspace I of L such that [x,y]$\in I$ for all x $\in I$ and y $\in L$

An important example of an ideal is the centre of L defined by
$Z(L) \equiv \{x \in L : [x, y] = 0 \text{ for all } y \in L\}$

We also have a notion of a homomorphism. A *Lie Homomorphism* is defined as the map $\phi : L_1 \to L_2$ such that

$$\phi([x, y]) = [\phi(x), \phi(y)] \tag{2}$$

An important example of Lie homomorphism will be the *adjoint homomorphism* defined as such

$$ad : L \to gl(L) \text{ by } (adx)(y) \equiv [x, y] \tag{3}$$

If $\phi : L_1 \to L_2$ is a homomorphism, then $ker(\phi)$ is an ideal of L_1 and the image of ϕ, $im(\phi)$, is a lie subalgebra of L_2 because if $x, y \in L_1$ and $x, y \in ker(\phi)$ then $\phi([x, y]) = [\phi(x), \phi(y)] = 0$. $\phi(x) = \phi(y) = 0$.
Now let there be an element $z \in L$ then $\phi([z, y]) = [\phi(z), \phi(y)] = 0$. So $ker\phi$ is an ideal of L_1.

3 Ideals and Homomorphisms

Ideals play a similar role that normal subgroups play in group theory. In other words we can use ideals to construct other types of lie algebras in the same way that we used normal subgroups to construct quotient groups. Suppose I and J are ideals of a lie algebra L, Then the following are true:

1. $I \bigcap J$ is an ideal, since we already we that $I \bigcap J$ is a subspace of L. So we need to check that for $x \in L$ and $y \in L$, $[x, y] \in I \bigcap J$, but the result follows quickly since I and J are both independently ideals of L.

2. I + J $\equiv \{x + y : x \in I, y \in J\}$ is an ideal. Let z $\in L$, consider [z,(x+y)] = [z,x] + [z,y] = a + b with $a \in J$ and $b \in J$. So a +b $\in I + J$ making I +J an ideal.

3. Product of ideals, [I,J] : Span $\{[x, y] : x \in I, y \in J\}$ is an ideal. To prove we start with the Jacobi Identity. $[u, [x, y]] + [x, [y, u]] + [y, [u, x]] = 0 \implies [u, [x, y]] = [x, [u, y]] + [[u, x], y] x \in I, y \in J$ and $u \in L$. So $[u, y] \in J$ since J is an ideal therefore $[x, [u, y]] \in [I, J]$ by the definition, similar argument applies for $[[u, x], y]$. Now a general element,t, of [I,J] is a linear combination i.e $t = \sum c_i[x_i, y_i]$ where c_i are scalars from the field and $x_i \in I, y_i \in J$. Now pick and element $u \in L$ and consider $[u, t] = [u, \sum c_i[x_i, y_i]] = \sum c_i[u, [x_i, y_i]]$. But $[u, [x_i, y_i]] \in [I, J]$ and so the sum is . So in summary we have that $[u, t] \in [I, J]$

A special construction occurs if we take I=J=L. We write [L,L] = $L^{'}$ and call it the derived algebra of L.

4 Quotient Algebras

We may consider the cosets of the ideal defined as follows $z + I = \{z + x : x \in I\}$ for $z \in L$ so the quotient vector space is $L/I = \{z + I : z \in L\}$. The lie bracket on L/I may be defined by $[w + I, z + J] := [w, z] + I$ for $w, z \in L$. The lie bracket above is bilinear i.e $[w+I, (u+I)+(v+I)] = [w+I, u+I]+[w+I, v+I]$ and $[w + I, (u + I) + (v + I)] = [w, u] + I + [w, u] + I$. Same argument applies for the left side. We also have that $[w + I, w + I] = [w, w] + I = I$. Jacobi Identity is also is satisfied i.e :

$$
\begin{aligned}
&[[u + I, [v + I, w + I]]] + [(v + I), [(w + I), (u + I)]] + [(w + I), [(u + I), (v + I)]] = \\
&[(u + I), [v, w] + I] + [(v + I), [w, u] + I] + [(w + I), [u, v] + I] = \\
&[u, [v, w]] + I + [v, [w, u]] + I + [w, [u, v]] + I = \\
&[u, [v, w]] + [v, [w, u]] + [w, [u, v]] + I = I
\end{aligned}
$$

We can motivate why we consider the cosets of ideals rather than any old sub-algebra. The main motivation is that we want the lie bracket defined on the quotient algebra to be well-defined in other words we do not want the answer to depend on the representative of the coset we chose. Consider the following: We have defined $[x + I, y + I]$ to be $[x, y] + I$. But suppose we choose another representative for each coset and perform the bracket i.e $[(x+j)+I,(y+k)+I]$ for $j,k \in I$, we still want the answer to be $[x,y]+I$ since x and $x+j$, y and $y+k$ are in the same cosets respectively. We now do the computation

$$\begin{aligned}[(x+j)+I,(y+k)+I] &= [(x+j),(y+k)]+I \\ &= [x,y]+[j,y]+[x,k]+[j,k]+I\end{aligned}$$

It should be clear that the last commutator is in I and just the first commutator is the result we want. It then follows that $[y,j],[x,k] \in I$ but this makes I an ideal.

We now see the analogue of the three isomorphism theorems we saw for groups in the context of lie algebras.

4.1 Isomorphism Theorems

1. Let $\phi : L_1 \to L_2$ be a homomorphism of Lie algebras. Then $ker\phi$ is an ideal of L_1 and $im\phi$ is a subalgebra of L_2 and $L_1/ker\phi \simeq im\phi$
2. If I and J are ideals of a lie algebra, then $(I+J)/J \simeq I/(I \bigcap J)$
3. Suppose that I and J are ideals of a lie algebra L such that $I \subseteq J$. Then J/I is an ideal of L/I and $(L/I)/(J/I) \simeq L/J$

5 Solvable Lie Algebras

We take an ideal of I of a lie algebra L and ask when the factor or quotient algebra L/I is abelian.

Lemma 5.1 *Suppose I is an ideal of L. Then L/I is abelian iff I contains the derived algebra* $L^{'}$

Proof The algebra L/I is abelian iff for all x,y $\in L$ we have [x+I, y+I]= [x,y] + I = I or $\forall x, y \in L$ we have $[x,y] \in I$. Since I is a subspace of L, this holds iff the space spanned by the brackets [x,y] is contained in I, $L^{'} \subseteq I$.

This argument says that the derived algebra $L^{'}$ is the smallest ideal of L that has an abelian quotient. By the same argument then derived algebra $L^{'}$ has a smallest ideal whose quotient is abelian. We denote this smaller derived algebra as $L^{(2)}$. The argument goes on iteratively. We can define the derived series of L to be the series with the terms $L^{(1)} = L^{'}$ and $L^{(k)} = [L^{(k-1)}, L^{(k-1)}]$. Then $L \supseteq L^{(1)} \supseteq L^{(2)} \supseteq L^{(3)} \dots$

Definition The lie algebra L is said to be solvable if for some m ≥ 1 we have $L^{(m)} = 0$

As a consequence the Heisenberg algebra is solvable but $\mathbf{sl}(2, C)$ is not solvable.

If L is solvable, then the derived series of L provides us with an "approximation" of L by a finite series of ideals with abelian quotients. This works the other way round.

Lemma 5.2 *If L is a lie algebra with ideals* $L = I_0 \supseteq I_1 \supseteq I_2 \supseteq I_3 \ldots I_{m-1} \supseteq I_m = 0$ *such that* I_{k-1}/I_k *is abelian for 1* $1 \leq k \leq m$*, then L is solvable.*

Proof Key idea: Show that $L^{(k)}$ is contained in I_k for k between 1 and m. Putting k=m will then give $L^{(m)} = 0$. L/I_1 is abelian, we know that $L^{'} \subseteq I_1$ For the inductive step, we suppose $L^{(k-1)} \subseteq I_{k-1}$ where $k \geq 2$. By construction I_{k-1}/I_k is abelian, this means that $[I_{k-1}, I_{k-1}]$ must be contained in I_k(By Lemma 3.5.1). But $L^{(k-1)}$ is contained in I_{k-1} by our inductive hypothesis so we deduce that $L^{(k)} = [L^{(k-1)}, L^{(k-1)}] \subseteq [I_{k-1}, I_{k-1}]$ and hence $L^{(k)} \subseteq I_k$. QED

This proves that if $L^{(k)}$ is non-zero then I_k is also non-zero. Hence the derived series may be thought of as the fastest descending series whose successive quotients are abelian.

Lie algebra homomorphism are linear maps that preserve Lie Brackets, and so one would expect that they preserve the derived series. Suppose that $\phi : L_1 \to L_2$ is a surjective homomorphism of a lie algebras show that $\phi(L_1^{(k)}) = (L_2)^{(k)}$, we proceed by induction on k. We already have that $\phi(L_1) = L_2$.

So $\phi([L_1, L_1]) = [\phi(L_1), \phi(L_1)]$ by the property of homomorphisms and by assumption we now have that $[\phi(L_1), \phi(L_1)] = [L_2, L_2]$ $\therefore$ we have that $\phi : L_1^{'} \to L_2^{'}$. The inductive step is to assume that $\phi(L_1^{(k-1)}) = (L_2^{(k-1)})$ and consider the derived algebra of $L_1^{(k-1)}$. $\phi([L_1^{(k-1)}, L_1^{(k-1)}]) = [\phi(L_1^{(k-1)}), \phi(L_1^{(k-1)})]$. This is equal to $[L_2^{(k-1)}, L_2^{(k-1)}]$ by the inductive step. Our desired result therefore follows since $\phi([L_1^{(k-1)}, L_1^{(k-1)}) = \phi(L_1^{k}) = [L_2^{(k-1)}, L_2^{(k-1)}] = (L_2)^{k}$. QED.
It turns out that if L is a lie algebra then

1. if L solvable, then every subalgebra and every homomorphic image of L is solvable
2. Suppose that L has an ideal I such that I and L/I are solvable. Then L is solvable.
3. If I and J are solvable ideals of L then I+J is a solvable ideal of L.

Theorem 5.3 *Let L be a finite dimensional Lie algebra. There is a unique solvable ideal of L containing every solvable ideal of L.*

Proof Let R be a solvable ideal of the largest possible dimension. We know that if I and J are solvable ideals then I+J is solvable. Let I be a solvable ideal. We have that R+I as solvable this means that R $\subseteq R+I$ and therefore dim(R) $\leq$ dim(R+I). But we chose R to have the largest possible dimension and therefore dim(R)= dim(R+I) and hence R = R+I so I $\subseteq R$. The largest solvable ideal is called the *radical of L* and is denoted as *rad* L QED.

The notion of a radical of L suggests the following definition.

Definition A non-zero finite dimensional Lie Algebra L is said to be *semisimple* if it has no nonzero solvable ideals for equivalently rad L =0.

An example is $\mathbf{sl}(2, C)$ has non-trivial ideals so it is semisimple. If L is a lie algebra, then the factor algebra L/(rad L) is semisimple. This makes sense since rad +L is the unique solvable ideal that contains all other solvable ideals. Therefore if we mod out by that we are left with a factor algebra that has no non-zero solvable ideals.

6 Some Representation Theory

Purpose : Examine the ways in which an abstract Lie Algebra can be viewed concretely as a subalgebra of the endomorphism algebra of a finite dimensional vector space.

Definition Let L be a lie algebra over a field F. A representation of L is a lie algebra homomorphism $\phi : L \to \mathbf{gl}(V)$ where V is a finite dimensional vector space over F.

Suppose $\phi : L \to \mathbf{gl}(V)$ is a representation. The image of ϕ is a lie subalgbera of $\mathbf{gl}(V)$ and the kernel of ϕ is an ideal of L.

Thus in general we loose some information when we work with ϕ. But when the kernel is zero then the map is one to one and information is not lost. The representation is then said to be **faithful**.

Examples:

1. $ad : L \to \mathbf{gl}(L) : (ad_x)y = [x, y]$. This provides a representation of L with V=L. This is known as the adjoint representation. The kernel of the adjoint representation is Z(L). Hence the adjoint representation is faithful when the center of L is zero.

Consider the adjoint representation of $\mathbf{sl}(2, C)$. Show that with respect to basis (h,e,f) ad_h is the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

The basis for $\mathbf{sl}(2, C)$ is $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and note the following commutation relations $[h, f] = -f, [h, e] = 2e, [e, f] = h$. From these the result follows.

2. Suppose that L is a lie subalgebra of $\mathbf{gl}(V)$. The inclusion map $L \to \mathbf{gl}(V)$ is trivially a lie algebra homomorphism. The corresponding representation is knows as the **natural representation**

3. Every Lie algebra has a trivial representation. To define this representation, take V=F and define $\phi = 0$ for all $x \in L$

6.1 Modules for Lie Algebras

Suppose that L is a lie algebra over a field F. A _lie_ _module_ for L, or alternatively an L-module is a finite dimensional F-vector space V together with a map

$$L \times V \to V \qquad (x, v) \longmapsto x.v \tag{4}$$

satisfying the conditions

1. $(\lambda x + \mu y) = \lambda(x.v) + \mu(y.v)$
2. $x.(\lambda u + \mu w) = \lambda(x.u) + \mu(x.w)$
3. $[x, y].v = x(y.v) - y.(x.v)$

$\forall x, y \in L, v, w \in V$ and $\lambda, \mu \in F$

The first and second properties imply that the map $(x, v) \longmapsto x.v$ is a bilinear map and the second implies that the map $v \longmapsto (x.v)$ is a linear endomorphism of V, so elements of L act on V by linear maps.

6.2 Submodules and Factor Modules

Suppose that V is a lie module for the Lie Algebra L. A submodule of V is a subspace of W of V which is invariant under the action of L. i.e for each $x \in L, w \in W$, we have $x.w \in W$. In the language of representation, submodules are known as sub-representations.

Examples:
1. Let L be a lie algebra, we may make L into an L-module via the adjoint representation. The submodules of L are exactly the ideals of L.

Proof $x \in L, y \in L$, our map $L \times V \to V$ will be defined here as $L \times L \to L$ and will be the lie bracket. L is playing both the role of a lie algebra and a vector space so that $(x, y) \longmapsto [x, y] \in L$ by $(ad_x)y$. The subspaces that are invariant under this map are the ideals

2. Let $L = b(n, F)$ be the lie algebra of $n \times n$ upper triangular matrices and let V be the natural L-module, so by definition $V = F^n$ and the action of L is given by applying matrices to column vectors. Let $e_1, e_2 \dots e_n$ be the standard basis for F^n. For $1 \leq r \leq n$. Let $W_r = span\{e_1, \dots e_r\}$, W_r is a submodule of V.

3. $L \to$ complex solvable Lie algebra. Suppose $\phi : L \to \mathbf{gl}(V)$ is a representation of L. As ϕ is a homomorphism, $im\phi$ is a solvable sub-algebra of $\mathbf{gl}(V) \implies$ V has a one dimensional sub-representation.

Suppose that W is a submodule of the L-module V. We can give the quotient vector space V/W the structure of an L-module by setting

$$x.(v + W) := (x.v) + W \qquad \text{for } x \in L \text{ and } v \in V$$

We call this module the quotient or factor module V/W. For an example of a quotient module, suppose I is an ideal of the Lie Algebra L. The factor module L/I becomes an L-module via

$$x.(y+I) := (ad_x)y + I = [x, y] + I$$

Looking at it differently, L/I is a lie algbera with Lie bracket given by $[x+I, y+I] = [x, y] + I$. So regarded as a L/I-module, the factor module L/I is the adjoint representation of L/I on itself.

6.2.1 L-module Homomorphisms

Let L be a lie algebra and let V, W be L-modules. An L-module homomorphism from $V \to W$ is a linear map $\theta : V \to W$ such that $\theta(x.v) = x.\theta(v), \forall v \in V, x \in L$.
Let $\phi_V : L \to \mathbf{gl}(V)$ and $\phi_W : L \to \mathbf{gl}(W)$ be representation corresponding to V and W. In the language of representation theory, the condition becomes $\theta \circ \phi_V = \phi_W \circ \theta$. Because we have vector spaces and homomorphisms lying around there also analogues for the three isomorphism theorems for lie modules.

Concretely, we can give an example. Suppose we have a one dimensional abelian lie algebra L, spanned by x. We can find a representation for it $f \in \mathbf{gl}(V)$. But suppose we find another representation of it $g \in \mathbf{gl}(W)$ and we further suppose that there is a homomorphism θ from V to W. If this homomorphism turns out to be an isomorphism then we know from the way lie module homomorphisms work that f and g will be equivalent iff $\theta f = g\theta$. An explicit example is diagonalizing a matrix.

6.2.2 Schur's Lemma

A lie module V is said to be irreducible, or simple, it it is non-zero and t has no sub-modules other than $\{0\}$ and V. The L-module V is completely reducible if it can be written as a direct sum of irreducible L-modules; i,e $V = S_1 \bigoplus S_2 \ldots \bigoplus S_k$ where each S_i is an irreducible L-module. Suppose that S and J are irreducible Lie modules and that $\theta : S \to T$ is a non-zero sub-module homomorphism. Then $im\theta = J$. Similarly $\ker\theta$ is a proper sub-module of S, so $\ker\theta = 0$.It follows that θ is an isomorphism from S to J, so there are no non-zero homomorphisms between non-isomorphic irreducible modules. We now consider a homomorphism from a lie-module to itself.

Theorem 6.1 *(Schur's Lemma:) Let L be a complex Lie algebra and let S be a finite-dimensional irreducible L-module. A map $\theta : S \to S$ is an L-module homomorphism iff θ is a scalar multiple of the identity transformation i.e $\theta = \lambda I$ for some $\lambda \in \mathbb{C}$*

Proof "If" direction is simple. The "only if" direction is non-trivial. Suppose $\theta : S \to S$ is a L-module homomorphism, then θ is a linear map of a complex vector space, and so it must have an eigenvalue, say λ. Now $\theta - \lambda I$ is also L-module homomorphism. The kernel of this map contains the $\lambda-$ eigenvector for θ, and so it is a non-zero submodule of S. As S is irreducible, S $= \ker(\theta - \lambda I)$; that is $\theta = \lambda I$

7 The representation theory of sl(2,C)

In this section we concentrate on the representation theory of $\mathbf{sl}(2, C)$ because it lays out the ground work and the basic ideas for the representation theory of semi-simple groups in general but from a physics point of view it can be used to study the representation theory of SU(2) which is used to study angular momentum.

For our discussion we use the following basis introduced earlier: $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ We are going to construct irreducible representations of $\mathbf{sl}(2, C)$ by considering the vector space of polynomials in two variables X,Y with complex coefficients, $\mathbb{C}[X, Y]$. Let V_d be a particular subspace consisting of homogeneous polynomials i.e for each integer $d \geq 0$, let V_d consist of polynomials in X and Y of degree d. So,

1. $V_0 \to$ one dimensional constant polynomials
2. $V_d \to$ d+1 $(d > 0)$ dimensional has the following monomials as a basis $\{X^d, X^{d-1}Y, \dots XY^{d-1}, Y^d\}$

We make V_d into a $\mathbf{sl}(2, C)$-module by specifying a lie algebra homomorphism $\phi : \mathbf{sl}(2, C) \to \mathbf{gl}(V_d)$

$$\begin{aligned} \phi(e) &:= X\frac{\partial}{\partial Y} \\ \phi(f) &:= Y\frac{\partial}{\partial X} \\ \phi(h) &:= X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y} \end{aligned}$$

Note that $\phi(h)(X^aY^b) = (a - b)X^aY^b$ so $\phi(h)$ acts diagonally on V_d with respect to our chosen basis.

Theorem 7.1 *ϕ is a representation of* $\mathbf{sl}(2, C)$

Proof By construction it is linear all we have to check is that it obeys the commutation relations.
We show $[\phi(e), \phi(f)] = \phi([e, f]) = \phi(h)$.
$[\phi(e), \phi(f)]X^aY^b = \phi(e)\phi(f)X^aY^b - \phi(f)\phi(e)X^aY^b = \phi(e)Y^{b+1}X^{a-1}a - \phi(f)X^{a+1}Y^{b-1}b = X^aY^ba(b+1) - X^aY^b(a+1)b = X^aY^b(a(b+1) - (a+1)b) = X^aY^b(a-b) = \phi(h)X^aY^b$

We check action of X^d
$[\phi(e), \phi(f)](X^d) = \phi(e)\phi(f)X^d - \phi(f)\phi(e)X^d = \phi(e)dX^{d-1}Y - \phi(f)(0) = dX^d$
We check $[\phi(h), \phi(e)] = \phi([h, e]) = \phi(2e) = 2\phi(e)$
$[\phi(h), \phi(e)](X^aY^b) = \phi(h)(\phi(e)(X^aY^b) - \phi(e)\phi(h)(X^aY^b) = \phi(h)(bX^{a+1}Y^{b-1}) - \phi(e)((a-b)X^aY^b) = b((a+1)-(b-1)X^{a+1}Y^{b-1} - (a-b)bX^{a+1}Y^{b-1} = 2bX^{a+1}Y^{b-1}$

This is the same as 2 $\phi(e)(X^aY^b)$.Separate verification is needed for b=0 and a=d. Same can be done for $[\phi(h), \phi(f)] = -2\phi(f)$ $\square$

There is also a matrix representation in the basis $X^d, X^{d-1}Y, \ldots$ of V^d

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ldots & d \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} = \phi(e)$$

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ d & 0 & \ldots & 0 & 0 \\ 0 & d-1 & \ldots & \vdots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix} = \phi(f)$$

$$\begin{pmatrix} d & 0\ldots & 0 & 0 & 0 \\ 0 & d-2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \ldots & -d+2 & 0 \\ 0 & 0 & \ldots & 0 & -d \end{pmatrix} = \phi(h)$$

A $\mathbf{sl}(2, C)$ -submodule of V^d generated by any particular basis element X^aY^b contains all the basis elements and so is all of V^d, this can be proved.

Theorem 7.2 *The* $\mathbf{sl}(2, C)$*module* V^d *is irreducible*

Proof Suppose U is a non-zero $\mathbf{sl}(2, C)$ -submodule of V^d. Then $h.u \in U$ for all $u \in U$. Since h acts diagonalisably on V^d, it also acts diagonalisably on U so there is an eigenvector of h which lies in U. All eigenspace of h on V^d are one - dim'l and each eigenspace is spanned by one monomial X^aY^b, so the submodule U must contain some monomial but by the above theorem contains all the bases for V^d and so $U = V^d$ $\square$

7.0.1 Classifying the Irreducible sl(2,C) modules

Lemma 7.3 *Suppose V is an $\mathbf{sl}(2, C)$ -module and $v \in V$ is an eigenvector of h with eigenvalue λ*
i) Either $e.v$ = 0 or $e.v$ is an eigenvector of h with eigenvalue $\lambda + 2$
ii) Either f.v=0 or f.v is an eigenvector of h with eigenvalue $\lambda - 2$

Proof V is a representation of $\mathbf{sl}(2, C)$, so we have $h.(e.v) = e.(h.v) + [h, e].v = e(\lambda)v + 2e.v = (\lambda + 2)e.v$. The calculation for f.v follows similar steps. $\square$

Lemma 7.4 *Let V be a finite dimensional $\mathbf{sl}(2, C)$ module. Then V contains an eigenvector w for h such that $e.w = 0$*

Proof Since $\mathbb{C}$ is an algebraically closed field, the linear map $h : V \to V$ has at least one eigenvalue and so at least one eigenvector. Let $h.c = \lambda v$. Consider the vectors: $v, e.v, e^2v, e^3v, \ldots$. If these vectors are no zero then by the previous lemma we have an infinite sequence of h-eigenvectors with distinct eigenvalues. Eigenvectors with distinct eigenvalues are linearly independent and so V is infinite dimensional which is a contradiction since V is finite dimensional. This means that there must be some $kgeq0$ such $e^k.v \neq 0$ but $e^{k+1}.v = 0$. We set $w = e^k.v$ then $h.w = (\lambda + 2k)w$ and $e.w = 0\square$

We now proceed to our main result, in which we will classify the irreducible representations by creating an isomorphism with V^d and its irreducible representations.

Theorem 7.5 *If V is finite-dimensional irreducible $\mathbf{sl}(2, C)$ module, the V is isomorphic to one of the V^d*

Proof Key idea: Produce a basis that spans V and then construct the isomorphism to V^d.
<u>Step 1</u>
So by lemma 3.7.4 V has an h-eigenvector w such that $e.w = 0$.Suppose that $h.w = \lambda w$. So we consider the sequence of vectors $w, fw, f^2w, f^3, w \ldots$. We claim that there exists a d such that $f^d w \neq 0$ but $f^{d+1}w = 0$. This follows by the previous theorem. Now w, fw, f^2w, f^dw form a subspace of V and since all have distinct eigenvalues, it follows that they are linearly independent. By construction this subspace is invariant under the action of h and f. We show that it is invariant the action of e by induction i.e $ef^kw \in$ span $\{f^iw : 0 \leq i < k\} = U$. Note that for $k = 0$ we have that $ew = 0$, so for the indcutive step we note that $e(f^k)w = (fe + h)(f^{k-1})w$. By the inductive hypothesis we have that $e(f^{k-1}) \in U$ so $fef^{k-1}w \in U$ and $hf^{k-1} \in U$. This finishes the inductive step. Now V is irreducible so in fact U = V
<u>Step 2</u>
We now proceed to produce the isomorphism. We have that V is the span of $\{w, fw, \ldots f^dw\}$ and V^d has the basis $\{X^d, fX^d, \ldots f^dX^d\}$. Notice that the eigenvalue of h on f^kw is the same eigenvalue of h ron f^kX^d. The homomorphism which will later turn into an isomorphism must map h-eigenvectors to h-eigenvectors. So define it to be

$$\psi(w) = X^d$$

and then define ψ by

$$\psi(f^k w) := f^k X^d \tag{5}$$

This defines a vector space isomorphism which commutes with the action of f and h. We need to show it commutes with the action of e. Again we proceed by induction similar to that in step 1. So for $k = 0$ we have that $\psi(ew) = 0$ and $e\psi(w) = 0, eX^d = 0$. For the inductive step, $\psi(ef^k w) = \psi((fe + h).(f^{k-1}w)) = f\psi(ef^{k-1}w) + h\psi(f^{k-1}w)$ by the inductive step we can take the e out and obtain $(fe + h)\psi(f^{k-1}w) = ef\psi(f^{k-1}w) = e\psi(f^k w)\Box$

Corollary 7.6 *If V is a finite-dimensional representation of $\mathbf{sl}(2, C)$ and $w \in V$ is an h-eigenvector such that $ew = 0$ then $hw = dw$ for some non-negative integer d and the submodule of V generated by w is isomorphic to V^d*

Proof To prove that the eigenvalue is d, remember that in our chosen basis h is diagonal and therefore we can take the trace i.e $\lambda + \lambda - 2 + \ldots (\lambda - 2d) = (d+1)\lambda - (d+1)d$. But remember that the trace of h is zero so the $\lambda = d$. Now apply step 2 to get the required results.

The vector considered in the corollary is known as the heighest weight vector and its associated eigenvalue with respect to h is known as the highest weight.