Let X and Y be vector fields on a manifold M^n and let $\phi(t) = \phi_t$ be the local flow generated by X. We compare the vector $Y_{\phi_t x}$ at that point with the results of pushing Y_x to the point $\phi_t x$ means of the differential ϕ_{t*}. The **Lie Derivative** of Y with respect to X is defined as

$$[\mathcal{L}_X Y]_x : \lim_{t \to 0} [\frac{Y_{\phi_t x} - \phi_{t*} Y_x}{t}] \tag{1}$$

Alternatively

$$\begin{aligned} [\mathcal{L}_X Y]_x &= \lim_{t \to 0} \phi_{-t*} [\frac{\phi_{t*} Y_{\phi_t x} - Y_x}{t}] \\ &= \lim_{t \to 0} [\frac{\phi_{-t*} Y_{\phi_{t*} x} - Y_x}{t}] \end{aligned}$$

For second step we used ϕ_{0*} is identity. We next use Hadamard's Lemma to get a very useful form for the Lie Derivative. We simply state but offer no proof

Lemma 0.1 ***Hadamards's Lemma***: *Let f be a continuously differentiable function defined in a neighborhood U of x_0. Then for sufficiently small t, there is a function g_t continuously differentiable in t and point x in U such that:*

$$g_o(x) = X_x(f)$$

and

$$f(\phi_t x) = f(x) + t g_t(x)$$

Using this lemma we now find a better(more pleasing form) of the derivative. One might worry about whether the limit in the lie derivative exists but Hadamard's lemma takes care of that for us.

$$\begin{aligned} [\mathcal{L}_X Y](f)] &= \lim_{t \to 0} [\frac{Y_{\phi_t x} - \phi_{t*} Y_x}{t}](f) \\ &= \lim_{t \to 0} [\frac{Y_{\phi_t x}(f) - Y_x(f \circ \phi_t)]}{t} \\ &= \lim_{t \to 0} [\frac{Y_{\phi_t x}(f) - Y_x(f + t g_t)]}{t} \text{ by Hadamard's lemma} \\ &= \lim_{t \to 0} [\frac{Y_{\phi_t x}(f) - Y_x(f)]}{t} - \lim_{t \to 0} Y_x(g_t) \\ &= X_x\{Y(f)\} - Y_x(g_0) \text{ definition of lie derivative} \\ &= X_x\{Y(f)\} - Y_x\{X(f)\} \text{ by Hadamard's Lemma} \end{aligned}$$

We therefore arrive at an agreeable expression namely

$$\mathcal{L}_x Y = [X, Y] \tag{2}$$

The above is obviously the lie-bracket.

It is useful to develop the concept of the lie derivative from the concept of lie transport which is a kind of flow of vector field. We take a step back on concentrate on local flow of a vector field in order to introduce 'lie' flow.

1 Local flow of a vector field

A vector field "tears up" a manifold into a system of integral curves. If each point $p \in M$ moves a parametric distance t along "its own" integral curve, we get a map.

$$\Phi_t : M \to M \qquad P := \gamma(t_0) \longmapsto \gamma(t + t_0)$$

which is called the local flow generated by the field V. This map crucially does not depend on the value of the parameter to which we assign to P. Why?

Suppose we have an integral curve $\gamma(t)$ for vector field V. What is the most general re-parametrization such that $\gamma(\sigma(t))$ is still an integral curve for V? We know that $\dot{\gamma} = V$ and $\gamma(\hat{\sigma(t)})$. We want $\gamma(\dot{\hat{\sigma(t)}}) = \dot{\gamma} = V \implies \gamma(\dot{\sigma(t)})\dot{\sigma(t)} = \dot{\gamma} = V \implies \dot{\sigma(t)} = 1$ and $\sigma(t) = \int, dt \implies t + c$ where $c \in \mathbb{R}$

Now $\gamma(0) = P \in M$ and the same integral curve starting at $\hat{\gamma}(a) = Q \in M$ are the same integral curve for vector field V. From the above discussion we can re-parametrize with $\sigma(t) = t + a$ so $\gamma(t + a) = \hat{\gamma}(0) = \gamma(a)$ which we showed earlier had the same vector field V. The flow Φ_t has a composition property with respect to parameter t.

$$\Phi_{t+s} = \Phi_t \circ \Phi_s$$

We can prove this by using a re-parametrization we just proved was possible. We know $P := \gamma(t_0) \longmapsto \gamma(t_0 + t + s)$. This is on the left hand side of our composition property. On the right hand side we have $\Phi_t \circ \Phi_s = \Phi_t(\Phi_s(\gamma(t_0))) = \Phi_t(\gamma(t_0 + s)) = \gamma(t_0 + t + s)$. Clearly both sides are equal. This map can be thought of as

$$\begin{aligned} \Phi : M \times R &\to M \\ (x, t) &\longmapsto \Phi_t(x) \end{aligned}$$

and the composition property looks like

$$(x, t + s) \longmapsto \Phi_{t+s}(x)$$

If we have $\Phi_t : x^i \longmapsto x^i(t; x)$ then $V = \dot{x}^i(0; x)\partial_i$. We consider an example 1. Consider $\vec{r} \to e^{\lambda t}\vec{r}$

$$\begin{aligned} r = x\hat{i} + y\hat{j} + z\hat{k} \longmapsto e^{\lambda t}(x\hat{i} + y\hat{j} + z\hat{k}) \\ V^1 = x\lambda e^{\lambda t}\partial_x|_{t=0} = x\lambda\partial_x \\ V^2 = y\lambda e^{\lambda t}\partial_y|_{t=0} = y\lambda\partial_y \\ V^3 = z\lambda e^{\lambda t}\partial_z|_{t=0} = z\lambda\partial_z \end{aligned}$$

so $V = \lambda(x\partial_x + y\partial_y + z\partial_z)$

Fixed Points of the flow Φ_t (points on a manifold which do not move under the maps Φ_t for all values of t) coincide with zero points of the generating vector fields V, i.e (points $p \in M$ such that $V_p = 0$) Why? $V = \dot{x}^i(P_0; x)\partial_x$ for $\Phi_t : x^i \longmapsto x^i(t; x)$. Since $V_p = 0$ we have that $\hat{x}^i = 0 \implies x^i = P_0$ where $P_0 \in \mathbb{R}$ since $x^i := x^i(\gamma(t)) = x^i(t) = P_0 \implies \dot{x}^i = \dot{\gamma}(\gamma(t)) = 0 \implies \gamma(t)$ is a constant number and so there is no flow at P_0.

The map $\phi : (\mathbb{R}, t) \to Diff(M), t \longmapsto \Phi_t$ is a homomorphism of groups. Why? $\phi(t+s) = \Phi_{t+s}$ and $\phi(t) \circ \phi(s) = \Phi_t \circ \Phi_s$. But this is the composition property of the flow which we proved earlier. So indeed we can think of the ϕ as a homomorphism.

Now if $f : M \to N$ is a diffeomorphism and Φ_t a flow on M then

a) $\psi_t := f \circ \Phi_t \circ f^{-1}$ is a flow on N. We can see this by considering the following: $f^{-1} : N \to M, y \longmapsto f^{-1}y$, $\Phi_t : M \to M, \gamma(f^{-1}(y)) \longmapsto \gamma(f^{-1}(y) + t)$ and lastly $f : M \to N, \gamma(f^{-1}(y) + t) \longmapsto \gamma(y + f(t))$

b) On M Φ_t is generated by vector field V, $V^i = \dot{x}^i \partial_i$. So with the map f, we have $V_{\gamma(t)} = V^i(x(t))\partial_i \longmapsto V^i(y(t))\frac{\partial f}{\partial x}\frac{\partial}{\partial y}$. So the flow ψ_t on N is generated by $f_* V$

If $f : M \to M$ is a diffeomorphism and $\gamma(t)$ and integral curve of a field V which starts in $x \in M$. then the curve $f(\gamma(t))$ is the integral curve of the field $f_* V$ Why?

There is some field V' so that $\frac{df(\gamma(t))}{dt} = V' \implies \frac{df(\gamma(t))}{dt}|_{t=0} = V'_0 = f_*(\frac{d}{dt}|_{t=0}\gamma) = f_*(\dot{\gamma}) = f_* V$ since $\dot{\gamma} = V$

2 Lie Transport and Lie Derivative

Let V be a vector field on M and let $Phi_t : M \to M$ be the corresponding flow. Since is a diffeomorphism, it induces the mapping (pull back) of tensor fields of arbitrary type on M.

$$\Phi_t^* : \mathfrak{T}_q^p(M) \to \mathfrak{T}_q^p(M)$$

This is known as lie . Note the fields are transported a parametric distance t along the integral curves of the field V against the direction of the flow Φ_t.

Φ_t is a linear operator on $\mathfrak{T}_q^p(M)$ and $(\Phi_t^* a)(U, \dots V; \alpha \dots \beta) = a(\Phi_* U, \dots, \Phi_* V; \Phi_{-t}^* \alpha, \dots \Phi^* - t\beta)$. We look at an example with scalar functions

For Ψ on M, drawn in form of a point of a graph; i.e as a hypersurface $(x, \Psi(x)) \subset M \times \mathbb{R}$.

i) For $M = \mathbb{R}$, $V = \partial_x, \Psi(x) = e^{-x^2}$. What is $\Phi^*\Psi$

First we find the integral curves of the vector field which are straight lines since $\dot{x} = 1$. So $\Phi_t \Psi = \Psi \circ \Phi_t = e^{-(t+a)^2}$. Note that the gaussian has been shifted backwards, against the direction of the flow.

ii) $M = \mathbb{R}^2, V = -y\partial_x + x\partial_y, \Psi(x,y) = e^{-((x-2)^2+(y-3)^2)}$

We solve for the integral curves first. One finds the solutions to be $x(t) = B\cos t - A\sin t, y(t) = A\cos t + B\sin t$. Therefore $\Phi_t^*\Psi(x,y) = e^{-((B\cos t - A\sin t)-2)^2-((A\cos t+B\sin t)-3)^2}$

So in general the graph of a function $\Phi_t^*\Psi$ may be obtained from the graph of Ψ simply by a shift of the former by a parameter t against the integral curves of the field V.

Given Φ_t with vector field V, let $\gamma(\tau)$ be the integral curve of the field W. We can justify the idea that the integral curves $\Gamma(t)$ of the lie transported vector field Φ_t^*W are given simply as the Φ_{-t} images of the initial curves $\gamma(\tau)$. Since $\dot{\gamma} = V$ for some integral curve γ. We need to find what Φ_t^*W is, with the requirement that the right handside is the derivative of the integral curve. We note that $\Phi_t^*W = \Phi_{-t}W$ because $f^*W = f^{-1}W$Then we start with the following expression, which is then differentiated:

$$
\begin{aligned}
&\Phi_{-t} \circ \gamma(\tau) = \Gamma(\tau) \\
&\qquad \frac{d(\Phi_{-t} \circ \gamma(\tau))}{d\tau} = \frac{d\Gamma(\tau)}{d\tau} \\
&\qquad\qquad\qquad = \Phi_{-t}\frac{d\gamma(\tau)}{d\tau} = \frac{d\Gamma(\tau)}{d\tau} \\
&\qquad\qquad\qquad = \Phi_{-t}W = \frac{d\Gamma(\tau)}{d\tau} = \Phi_t^*W
\end{aligned}
$$

We consider two electrostatic fields $E_1 = E\partial_x, E_2 = \frac{k}{r^2}\partial_r$. We consider other vector field generating 3 different flows in three dimensional euclidean space namely, $V = \partial_x, U = \partial_y, W = y\partial_x - x\partial_y$.

What would the electrostatic fields look like after they were lie transported along the vector fields V, U and W? From the previous discussion, the transported field lines of the transported vector fields would be the images of intial field line under the map Φ_{-t}^i with Φ_t^i being the flow resulting from the vector fields V,U and W. The vector field lines of V are lines in the x direction, for U they are lines in the y directions and W are circles about the origin. So E_1 transported along V will give field lines shifted backward in x direction with along pointing in x direction, transported along U gives the initial field lines shifted downwards and transported along W gives field lines rotated clockwise by some angle. The same story applies for E_2.

It may happen that $\Phi_t^*A = A$ the A is said to be invariant. This happened above when E_1 was transported by ∂_x, ∂_y and E_2 by $y\partial_x - x\partial_y$. This need not happen in general for an arbitrary tensor field A. A convenient measure of this dependence is given by

$$\mathcal{L}_V A : \frac{d}{dt}|_{t=0}\Phi_t^* A \tag{3}$$

This definition is equivalent to the one given in 3.1. It will be shown to give the same results when applied to vector fields. It has the advantage that it is completely self contained. Note $\mathcal{L}_V A = 0$ happens iff A is invariant with respect to V.

2.1 Properties of the Lie Derivative

The lie derivative is a derivation of the tensor algebra which commutes wth contractions.For $\epsilon << 1$ we have

$$\begin{aligned}\Phi_\epsilon^* A &= A + \epsilon\frac{d}{dt}\Phi_\epsilon^* A|_{t=0} + O(\epsilon^2)\\ &= A + \epsilon\mathcal{L}_V A + O(\epsilon^2)\end{aligned}$$

1). On Functions

$$\begin{aligned}\mathcal{L}_V\Psi &= \frac{d}{dt}\Phi_t^* A|_{t=0}\\ &= \frac{d}{dt}\left(\Psi\circ x(t)\right)|_{t=0}\\ &= \frac{dy^j}{dt}\Psi,_j\frac{\partial x^i}{\partial y^j}|_{t=0}\\ &= J_j^i y^j\partial_j\Psi\\ &= V\Psi\end{aligned}$$

2. On Covectors which happen to be gradients of functions

$$\begin{aligned}\mathcal{L}_V(d\Psi) &= \frac{d}{dt}|_{t=0}\Psi_t^* d\Psi\\ &= \frac{d}{dt}d\Psi\circ\Phi_t\\ &= d\frac{d}{dt}|_{t=0}\Psi\circ\Phi_t\\ &= d\frac{d}{dt}\Phi_t^*\Psi|_{t=0}\\ &= d\mathcal{L}_V\Psi = d(V\Psi)\end{aligned}$$

3. general co-vectors fields $\alpha = \alpha_i(x)dx^i = a_i\otimes dx^i$

$$\begin{aligned}
\mathcal{L}_V(\alpha) &= \mathcal{L}_V(a_i \otimes dx^i) \\
&= \mathcal{L}_V(a_)) \otimes dx^i + a_i \otimes \mathcal{L}_V(dx^i) \\
&= V^j \alpha_{i,j} dx^i + \alpha^i d(\mathcal{L}_V x^i) \\
&= V^j \alpha_{i,j} dx^i + \alpha_i dV^i \\
&= \left(V^i \alpha_{i,j} + a_j V^j_{,i}\right) dx^i
\end{aligned}$$

4. on a co-ordinate frame field ∂_i

$$\mathcal{L}_V < dx^i, \partial_k > = < \mathcal{L}_V(dx^i), \partial_k > + < dx^i, \mathcal{L}_V(\partial_k) > \tag{4}$$

$$0 = V^i_{,j} < dx^j, \partial_k > + < dx^i, \mathcal{L}_V(\partial_k) > \tag{5}$$

$$-V^i_{,k} = dx^i(\mathcal{L}_V(\partial_k)) \tag{6}$$

$$-V^i_{,k}\partial_i = \mathcal{L}_V(\partial_k) \tag{7}$$

With the above calculations it is straight forward to calculate the lie derivative of a general tensor. So

$$\mathcal{L}_V A^{i\ldots j}_{k\ldots l} = V^m A^{i\ldots j}_{k\ldots l,m} + V^k_{,m} A^{i\ldots j}_{m\ldots l} + \ldots A^{i\ldots j}_{k\ldots m} V^l_{,m} - V^m_{,im} A^{m\ldots j}_{k\ldots l} \ldots - A^{i\ldots m}_{k\ldots l} V^m_{,jm} \tag{8}$$

In particular let us calculate the lie derivative of vector field.

$$\begin{aligned}
\mathcal{L}_V(W) &= V^m W^j_{,m} \partial_j - W^j V^m_{,j} \partial_m \\
&= V^m W^j_{,m} \partial_j - W^m V^j_{,m} \partial_j \\
&= (V^m W^j_{,m} - W^m V^j_{,m}) \partial_j \\
&= [V, W]
\end{aligned}$$

The lie derivative has been expressed in terms of the pull back of the flow, but the flow can also be expressed in terms of the lie derivative. How can this be done?

$$\begin{aligned}
\frac{d\Phi^*_t}{dt} &= \frac{d\Phi_{t+s}}{ds}|_{s=0} \\
&= \Phi^*_t \frac{d}{ds}|_{s=0} \Phi^*_s \\
&= \Phi^*_t \mathcal{L}_V
\end{aligned}$$

The formal solution to the above O.D.E is $\Phi^*_t = e^{t\mathcal{L}_V}$. We now try a simple calculation in the case when $M = \mathbb{R}[x], V = \partial_x$. $\Phi^*_t \Psi = e^{t\mathcal{L}_V}\Psi = \Psi(x) + t\mathcal{L}_V(\Psi) + \frac{t^2}{2}\mathcal{L}_V(\mathcal{L}_V)\Psi + \cdots = \Psi + t\Psi' + \frac{t^2}{2}\Psi'' + \cdots = \Psi(x+t)$. Note we agree with what we found when we applied the formal definition of lie transport to scalar function. The function has been translated backwards against the flow. Also if the flow is not defined globally then this formula is wanting.

3 Isometries and Conformal transformation, Killing Equations

Let γ be a curve on M, f a transformation of M(diffeomorphism $f : M \to M$) and $\hat{\gamma} = f \circ \gamma$, the curve transformed by f. Denote by $l[\gamma, g]$ the functional of the length of a curve on a manifold (M,g). i.e

$$l[\gamma, g] := \int_{t_1}^{t_2} dt \sqrt{g(\dot{\gamma}, \dot{\gamma})}$$

For the transformed curve one obtains for the length the following equation $l[f \circ \gamma, g] = l[\gamma, f^* g]$ Why? $\dot{\gamma} \longmapsto f_* \dot{\gamma} \implies \sqrt{g(\dot{\gamma}, \dot{\gamma})} \longmapsto \sqrt{g(f_* \dot{\gamma}, f_* \dot{\gamma})} = \sqrt{f^* g(\dot{\gamma}, \dot{\gamma})}$

If the length does not change then we demand that for $f : M \to M, f * g = g$. Such transformation are called Isometries. The isometries automatically preserve angles under which arbitrary curves intersect. Why? Let 2 curves intersect in $x \in M$ at an angle α and let vectors v, w be tangent(of any length) to the curves in x. Then under f we have $x \longmapsto f, v \longmapsto f_* v, w \longmapsto w$ and $\cos \alpha \longmapsto \cos \alpha' = \cos \alpha$ because $\cos \alpha = \frac{g(v,w)}{\sqrt{g(v,v)}\sqrt{g(v,v)}}$ and $\cos \alpha' = \frac{g(f_* v, f_* w)}{\sqrt{g(f_* v, f_* v)}\sqrt{g(f_* v, f_* v)}} = \frac{f^* g(v,w)}{\sqrt{f^* g(v,v)}\sqrt{f^* g(v,v)}} = \frac{g(v,w)}{\sqrt{g(v,v)}\sqrt{g(v,v)}} = \cos \alpha$.

But from the above work we see that a weaker condition is possible to preserve angles; namely if g is changed by a function.$\sigma : M \to \mathbb{R}, i.e f^* g = \sigma g$. Such transformations are called conformal transformation of a manifold. For $\sigma = constant$ we have homotheties and $\sigma = 1$ we have isometries.

Conformal transformations constitute a group because if $f^* g = \sigma g$ and $h^* g = \sigma' g$ for $f, hM \to M$ and $\sigma, \sigma' M \to \mathbb{R}$

$$(f \circ h)^* g = (h^* \circ f^*) g = \sigma \sigma' g$$

Obviously, homotheties are a subgroup and so are isometries.

A tool for finding all isometries which may be obtained by a smooth deformation of the trivial isometry is

1. Find all infinitesimal isometries $\Phi_\epsilon : M \to M$. In co-ordinates $x^i \longmapsto x^i + \epsilon \xi^i(x)$

2. Obtain the finite maps by iterations of the infinitesimal ones. In this way we get a whole one paramer group = flow of isometries $\Phi_t : M \to M$ with the generator of the flow being the vector field $\xi = \xi^i(x) \partial_i$

We now find the equations which specify ξ. Let $\Phi_t : M \to M$ be a one-parameter group(flow) of isometries, generated by a vector field ξ then we know

$\mathcal{L}_\xi g = 0$. So we calculate the lie derivative of the metric tensor.

$$\begin{aligned}\mathcal{L}_\xi\left(g_{ij}(dx^i \otimes dx^i)\right) &= \xi^k g_{ij,k} dx^i \otimes dx^j + g_{kj}\xi^i_{,k} dx^k \otimes dx^j + g_{ik} dx^i \otimes \xi^j_{,k} dx^k \\ &= \xi^k g_{ij,k} + g_{kj}\xi^i_{,k} + g_{ik}\xi^j_{,k} = 0\end{aligned}$$

This is an over determined system we have n unknown functions $\xi^1(x), \dots \xi^n(x)$ but we have $\frac{n(n+1)}{2}$. Given two solutions ξ, η of the killing equation, then both $\xi + \lambda\eta$ and $[\xi, \eta]$ are solutions. The reason is that these vector fields form a lie algebra. This lie algebra is of dimension at most $\frac{n(n+1)}{2}$. The fact that we have a sub-algebra, we can use this concept to find new solutions by taking all possible commutators. We do some examples next.

1. We find the killing vectors and corresponding flows for the euclidean plane. $\xi^1(x, y) = A(x, y), \xi^2(x, y) = B(x, y)$. We have $g_{ij} = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$

$$\begin{aligned}
&0 = \xi^k_{,i} g_{kj} + \xi^k_{,j} g_{ik} \\
&0 = \xi^1_{,i} g_{1j} + \xi^2_{,j} g_{2k} + \xi^1_{,j} g_{i1} + \xi^2_{,j} g_{i2} \\
i = 1, j = 1& \\
&0 = \xi^1_{,1} g_{11} + \xi^2_{,1} g_{21} + \xi^1_{,1} g_{11} + \xi^2_{,j} g_{12} \implies \frac{\partial A}{\partial x} = 0 \\
i = 1, j = 2& \\
&0 = \xi^1_{,1} g_{12} + \xi^2_{,1} g_{22} + \xi^1_{,2} g_{i1} + \xi^2_{,2} g_{12} \implies \frac{\partial B}{\partial x} = 0 \\
i = 2, j = 1& \\
&0 = \xi^1_{,2} g_{11} + \xi^2_{,2} g_{21} + \xi^1_{,1} g_{21} + \xi^2_{,1} g_{22} \implies \frac{\partial A}{\partial x} + \frac{\partial B}{\partial x} = 0 \\
i = 2, j = 2& \\
&0 = \xi^1_{,2} g_{12} + \xi^2_{,2} g_{22} + \xi^1_{,2} g_{21} + \xi^2_{,2} g_{22} \implies \frac{\partial B}{\partial x} = 0
\end{aligned}$$

All the above results imply $A(x, y) \to A(y), B(x, y) \to B(y)$ and $\frac{dA}{dy} = -\frac{dB}{dx} = constant$. the solutions for A and B are $A = -ky + x_0, B = kx + y_0$. So $\xi = A\partial_x + B\partial_y = k(-y\partial_x + x\partial_y) + x_0\partial_x + y_0\partial_y$. There are 3 independent solutions: $e_1 = -y\partial_x + x\partial_y, e_2 =$. So $\xi = A\partial_x + B\partial_y = k(-y\partial_x + x\partial_y) + x_0\partial_x, e_3 = \partial_y$.

We could use the killing equations in another way. We guess the form of ξ and solve for the most general g_{ij} that has these symmetries. This is used in solving complicated PDE. This strategy holds for $\mathcal{L}_A = 0$ for a general tensor field A.