1 K-dimensional distribution on a manifold

We a fix a k-dimensional subspace in each of the spaces

$$\mathcal{D}_x \subset T_x M \ \dim \mathcal{D}_x = k$$

If these subspaces depend smoothly on x, we say that a k-dimensional smooth distribution was defined on M. For the theory of connections, a distribution is given and we look for a $k-$ dimensional stratification on M.
Let L be an n-dimensional linear space and $\alpha \in L$ be a co-vector . We can check that the set W of vector which annihilate the covector i.e

$$W := \{w \in L| < \alpha, w >= 0\}$$

is actually an $(n-1)$ dimensional subspace of space L.
If $\alpha = e^1$ then the w that annihilate α will be those for which $w^1 \ = 0$. If $\alpha = a_1 e^1 + a_2 e^2$, then the w that annihilate will be those for which $w^1 a_1 = -w^2 a_2$. If $a_1 e^1 + a_2 e^2 + \ldots a_n e^n$ then the w that annihilate α will be those such that $w^1 a_1 = -(w^2 a_2 + w^3 a_3 + \ldots w^n a_n)$. Therefore there is one constraint and W has to be (n-1) dimensional.

Supposing β is another co-vector in L then their annihilation space turns out to be $(n-2)$ dimensional subspace of L. Doing a similar procedure as the paragraph first set that for the two co-vectors $\alpha = e^1, \beta = e^2$ and eventually we get two constraints namely $a_1 w^1 = -(w^2 a_2 + w^3 a_3 + \ldots w^n a_n)$ and $b_2 w^2 = -((w^1 b_1 + w^3 b_3 + \ldots w^n b_n)$. This process can be generalized to q linearly independent vector in L making W an $(n-q)$ dimensional space. If $e_a = (e_\alpha, e_i)$ is a basis in L which adapted to the subspace W i.e $e_\alpha \in W, \alpha = 1, \ldots k$ then the co-vector $e^i (i = k+1, \ldots n)$ define (as constraint 1 forms) the same subspace W. L^* has basis (e^α, e^i). Why? From the work above the annihilation space created by e^i will be $(n-(n-k))$ dimensional which be a k dimensional and so isomorphic to W. In-fact the subspace W is not changed if e_α are scrambled since we have $e'_\alpha \equiv A^\beta_\alpha e_\beta$ we can find B^i_j such that $B^i_j e^j = e'^i$ annihilate e'_α.

$A \in GL(k, \mathbb{R})$ or $B \in GL(n-k, \mathbb{R})$ gives a certain degree of freedom in fixing a subspace. This freedom may be reduced if we use an $(n-k)$ form instead of $(n-k)$ 1 forms.

What we have described is a way to get a k dimensional distribution. We say that vector field$V$ belongs to the distribution $\mathcal{D}$ if at each point of the domain $\mathcal{O}$ the value V_x of the field belongs to the subspace $\mathcal{D}_x$ given by the distribution. Let a k-dimensional distribution $\mathcal{D}$ in a domain $\mathcal{O}$ be given be vector fields e_α or alternatively by constraint 1 -forms θ^i then $< \theta^i, e_\alpha >= 0$ and $V \iff \mathcal{D}$
$V = V^\alpha(x) e_\alpha \iff < \theta^i, V >= 0$

Example

Consider a 2-dimensional smooth distribution $\mathcal{D}$ in $\mathbb{R}^3$ given by constraint 1 -form $\theta^3 \equiv \theta := dz + xdy - ydx \equiv dz + r^2 d\phi$

i) if $a, b, c \in \mathcal{F}(\mathbb{R}^3)$ and $V = a\partial_x + b\partial_y + c\partial_z$ then $V \in \mathcal{D} \iff c = ya - xb \iff V = a(\partial_x + y\partial_z) + b(\partial_y - x\partial_z)$ why?

$$< dz + xdy - ydx, a\partial_x + b\partial_y + c\partial_z >= 0 \iff$$
$$c + xb - ya = 0 \iff c = ya - xb \iff V = a(\partial_x + y\partial_z) + b(\partial_y - x\partial_z)$$

So we choose e_α fields as $e_1 = \partial_x + y\partial_z, e_2 = \partial_y - x\partial_z$.

The above distribution is interesting because it is non-integrable. First we introduce the *Frobenius criterion*

Theorem 1.1 *Frobenius Theorem: In terms of vector fields a distribution $\mathcal{D}$ is integrable iff the commutator of arbitrary vectors fields from $\mathcal{D}$ also belongs to $\mathcal{D}$ i.e $\mathcal{D}$ is integrable* $\iff \{U, V \in \mathcal{D} \implies [U, V] \in \mathcal{D}\}$

We can now show that the distribution introduced above is non-integrable since

$$\begin{aligned}[e_1, e_2](f) &= [\partial_x, \partial_y - x\partial_z] + [y\partial_z, \partial_y - x\partial_z](f) \\ &= -2\partial_z(f)\end{aligned}$$

$-2\partial_z \neq= ae_1 + be_2$ so $\mathcal{D}_x$ was not integrable. In terms of constraint 1-forms θ^i, Frobenius theorem says that a distribution $\mathcal{D}$ is integrable $\iff$ for arbitrary vectors $U, V \in \mathcal{D}$ there holds $d\theta^i(U, V) = 0$ i.e if the restriction of all 2-forms $d\theta^i$ to distribution $\mathcal{D}$ vanish. For our example $d\theta(e_1, e_2) = 2 \neq 0$

A constraint 1 -form θ of a 2 dimensional distribution in ordinary 3 dimensional euclidean space E^3 has the form $\theta = A \cdot dr$. The distribution $\mathcal{D}$ given by this 1 -form consists in each point of vectors which are perpendicular to A. So $i_B(A \cdot dr) = B \cdot A$ but since according to frobenius theorem we must have $\theta^i|_{\mathcal{D}}$ we have that $B \cdot A = 0$. We also have that $A \cdot (\nabla \times A) = 0$ Why? Frobenius $\implies d\theta = \sigma \wedge \theta$ where $\sigma = C \cdot dr$, $C \cdot dr \wedge Adr = (C \times A)dS = (\nabla \times A)dS$ but $d(C \times A)dS = div(C \times A)dV$. Let $G = C \times A$, so $A \cdot (\nabla \times A) = A \cdot G = 0$

2 Linear Connection and the Frame Bundle

Advantage: Contrary to what happened with describing the connection on a manifold where we needed to use local co-ordinates with the introduction of the frame bundle we can give the connection a global structure.

Consider the set LM of all frames $e(x)$ at all points x of a manifold M

$$LM := \bigcup_{x \in M} e(x)$$

This is endowed with a structure of a smooth manifold of dimension $(n+n^2)$ Why? Let x^i be local co-ordinates on $\theta \subset M$ and $e(x)$ a frame field defined on the same $\mathcal{O}$. Then for an arbitrary frame E in x we may write $E = e(x)y$ i.e $E_a = e_b y^b_a$ for $y \in GL(n, \mathcal{R})$ sox^i, y^b_a may serve as co-ordinates on $\mathcal{O} \subset M$

Define a map $\pi : LM \to M, e_a(x) \longmapsto x$. This is a smooth map with co-ordinate representation $(x^i, y^b_a) \longmapsto x^i$. For arbitrary x the pre-image $\pi^{-1}(x)$ is diffeomorphic to $GL(n, \mathbb{R})$. The above is called the *frame bundle*.

There is more structure, namely a natural action of the group $GL(n, \mathbb{R})$. If $A \in GL(n.\mathbb{R})$ then the map $R_A : LM \to LM, e \longmapsto R_A e = eA$ is a right action of $GL(n, \mathbb{R})$ on LM.

Note: This is different from the fiber. For the first $GL(n.\mathbb{R})$ we picked a frame at a point and applied y to to another frame. But this new action is applied on y i.e $R_A : (x^i, y^a_b) \longmapsto (x'^i, y'^a_b) = (x'^i, y^a_c A^c_b)$. This action is free, transitive and vertical i.e $\pi \circ R_A = \pi$

2.1 Connection form on LM

In order to describe a global connection on the manifold M, we need a covering of the manifold by open domains $\mathcal{O}_\alpha$ along with locally defined connection 1 forms; on each overlap $\mathcal{O} \bigcap \mathcal{O}'$ compatibility is ensured by

$$e' = eA \implies \hat{\omega}' = A^{-1}\hat{\omega}A + A^{-1}dA$$

Assuming $e' = eA$ then in the overlap region pulling back to the frame bundle we have that $\omega_{\mathcal{O}} = \omega_{\mathcal{O}'}$ since

$$\begin{aligned}
\omega_{\mathcal{O}'} &:= y'^{-1}(\pi^*\hat{\omega})y' + y'^{-1}dy') \\
&= (A^{-1}y)^{-1}(\pi^*(A^{-1}\omega A + A^{-1}dA)A^{-1}y + (A^{-1}y)^{-1}d(A^{-1}y) \\
&= y^{-1}A\left[\pi^* A^{-1}\hat{\omega}A + A^{-1}dA\right]A^{-1}y + y^{-1}A\left[dA^{-1}y + A^{-1}dy\right] \\
&= y^{-1}\pi^*\hat{\omega}y + y^{-1}d(AA^{-1})y + ydy \\
&= y^{-1}\pi^*\hat{\omega}y + y^{-1}dy
\end{aligned}$$

This means that there is a global connection on LM written as $\omega \equiv \omega^a_b E^b_a \in \Omega^1(LM, gl(n.\mathbb{R}))$. Making use of 1-forms $\hat{\omega}_\omega$ on M we can reconstruct ω on LM as follows

Let local section be defined as follows $\sigma : \mathcal{O} \to LM$ over the frame bundle $\pi : LM \to M$. These are in 1-1 correspondence with the frame fields i.e $\sigma(x) = e(x)$. Thi is done by letting $y^a_b = \delta^a_b$ so that $x^i \longmapsto (x^i, \delta^a_b)$. Now let $\omega \in \Omega^1(LM, gl(n, \mathbb{R}))$ be the connection on LM then $\hat{\omega} = \sigma^*\omega$. In $\pi^{-1}(\mathcal{O})$ we have $\omega = y^{-1}\pi^*\hat{\omega}y + y^{-1}dy$ but$y = I$ so $\omega = \pi^*\hat{\omega} \implies \sigma^*\omega = \sigma^* \circ \pi^*\hat{\omega} = (\pi \circ \sigma)^*\hat{\omega} = \hat{\omega}$

2.2 Properties of Linear Connection

1. $R^*\omega = Ad_{A^{-1}}\omega \equiv= A^{-1}\omega A$
First we derive $R^*_g e_i$.

$$
\begin{aligned}
R_g^* e_i &= R_{*g^{-1}} e_i \\
&= L_h E_i \\
&= L_h R_{*g^{-1}} E_i \\
&= L_{*h} L_{g^{-1}*} Lg * R_{g^{-1}*} E_i \\
&= L_{h*} L_{g^{-1}*} Ad_g E_i \\
&\equiv (Ad_g)^i_j e_i
\end{aligned}
$$

using $\delta^i_j =< e^i, e_j >=< R_g^* e^i, R_g^* e_i >$ we get $(Ad_{g^{-1}})^i_j e^j$.
The components of $\omega i.e E^b_a$ transform like e^i so that $R_A^* \omega = Ad_{A^{-1}} \omega$

2. Let ξ_C be the fundamental field of the action R_A which corresponds to a $C \in gl(n, \mathbb{R})$. $\xi_C = (yC)^a_b \partial^b_a \equiv C^a_b \xi_{E^a_b}$ where $\xi_{E^a_b} = y^c_b \partial^a_c$ and $\partial^a_c \equiv \frac{\partial}{\partial^c_a}$. Then $< \omega, \xi_C >= C$

3. $\mathcal{L}_{\xi_C} \omega = -ad_C \omega = -[C, \omega]$ (A proof of this is given later.)

2.3 Geometrical interpretation of a connection form: Horizontal distribution on LM

A *vertical distribution* $\mathcal{D}^v$ may be defined on LM so that the vertical subspace $Ver_e LM$ is declared to be the subspace which the distribution singles out in each tangent space.

$$\mathcal{D}^v_e := Ver_e LM \equiv Ker \pi_* \subset T_e LM$$

so that $W \in \mathcal{D}^v_e \iff \pi_* W = 0$ A general vector field on LM is given by $\xi = a\frac{\partial}{\partial x} + c^a_b \partial y^a_b$ so a general vertical vector field has the form

$$W = W^b_a(x, y) \partial y^a_b = W^b_a(x, y) \xi_{E^a_b} \tag{1}$$

It has dimension n^2 and is integrable since for any 2 $V, W \in \mathcal{D}^v_e$, $[W, V] \in \mathcal{D}^v_e$. If there is an action of a group G on a manifold M, an action on distributions on the manifold is naturally induced: if R_g shifts points, then R_{g*} shifts vectors and so

$$\mathcal{D}_x \longmapsto R_{g*} \mathcal{D}_x := (R_g \mathcal{D})_{xg} \tag{2}$$

It may happen that $\mathcal{D}$ is G-invariant $R_g \mathcal{D} = \mathcal{D}$ i.e the shifted subspace always happens to coincide with the subspace residing originally at the shifted point. Thus a vertical distribution on LM is $GL(n, \mathbb{R})$ invariant i.e for $A \in GL(n, \mathbb{R})$ $R_A \mathcal{D}^v = \mathcal{D}^v$ or $R_{A*}(\mathcal{D}^v_e) = \mathcal{D}^v_{eA}$. Locally we have that $R_{A*}(x, y^a_b) = (x, y^a_b A^b_c)$.

The construction of the vertical distribution needs no extra structure.

Let $\omega = \omega^a_b E^b_a$ be a connection form. The 1-forms w^a_b are linearly independent. Why? Let $k^a_b \omega^a_b = 0$ then $0 =< k^a_b \omega^b_a, \xi_C >= k^a_b C^b_a$ since $C \neq 0$ then $k^a_b = 0$ meaning ω^a_b are linearly independent.

Now $< \omega, V >= 0 \iff V \in \mathcal{D}^h$ defines a smooth n-dimensional distribution $\mathcal{D}^h$ on LM is defined; it is called the *horizontal distribution*. The subspace singled out (at each point $e \in LM$) by the distribution is called the *horizontal subspace*

$$Hor_e LM \equiv \mathcal{D}^h_e \subset T_e LM \tag{3}$$

Let $\mathcal{D}^h$ be the horizontal distribution on LM given by a connection form ω. We can check that if $v \in T_x M$, then at each point $e \in \pi^{-1}(x) \exists$ a horizontal lift i.e a unique vector $v^h \in T_e LM$ such that $\pi_* v^h = v$ for $v^h \in Hor_e LM$. We have $e = a\partial_i - y^a_b \partial^b_a e \in T_e LM$ with $\pi_* e = a\partial_i, a\partial_i \in T_x M$.

The distribution $\mathcal{D}^h$ is spanned by vector fields

$$H_i \equiv \partial_i - < \omega^a_b, \partial_i > y^b_c \partial^c_a \equiv \partial_i - < \omega^a_b, \partial_i > \xi_{E^b_a} \tag{4}$$

A general horizontal vector field V on LM may be written in the form

$$V \in \mathcal{D}^h \iff V = v^i(x,y) H_i \equiv V^i(x,y) \partial^h_i \tag{5}$$

The proof the statement follows by picking the ansatz $v^h = v^i \partial_i + v^a_b \partial^b_a$ using $< \omega^a_b, v^h >= 0$ we have $< \omega^a_b, v^h >= 0 =< \omega^a_b, v^i \partial_i > + < \omega^a_b, v^a_b \partial^b_a > \implies -v^i < \omega^a_b, \partial_i >= v^a_b \implies v^h = v^i \partial_i - v^i < \omega^a_b, \partial_i > \partial^b_a$. We also have the operation of the horizontal lift $v \longmapsto v^h$ is a linear isomorphism of the whole tangent space in x and the horizontal subspace in e since $v^i \partial_i \leftrightarrow v^i H_i$.

If a vector turns out to be at the same time horizontal and vertical, it is necessarily zero because if $e \in Lm$ and is in $Ver_e LM$ then $< \omega, \xi_C >= C$ but it is also in $Hor_e LM$ then $C = 0$. This means that

$$T_e LM = Ver_e LM \bigoplus Hor_e LM$$

The horizontal distribution $\mathcal{D}^h$ on LM is $GL(n, \mathcal{R})$ invariant i.e

$$R_A \mathcal{D}^h = \mathcal{D}^h \text{ or } R_{A*}(\mathcal{D}^h_e) = \mathcal{D}^h_{eA}$$

because we know that $R_A \omega = Ad_{A^{-1}} \omega = A^{-1} \omega A$ also for $v \in Hor_e LM$ we have $< \omega, v >= 0$ then $< \omega, R_{A*} v >=< \omega, Ad_{A^{-1}} >= A^{-1} < \omega, v > A = 0$

2.4 Horizontal distribution on LM and Parallel transport on M

Imagine we have a curve $\gamma(t)$ on M and a field of frames $e(t)$ on the curve. Such a field of frames then induces naturally a curve $\hat{\gamma}$ on LM, one assigns a frame $e_a(\gamma(t))$ to the parameter t, interpreted as a point on LM. We fix a frame E at a point of the curve γ. Making use of the connection, we generate an auto-parallel frame field $e^{\parallel}(t)$ on $\hat{\gamma}$. $\hat{\gamma}$ has the feature of horizontality i.e its tangent vector is horizontal at each point. (The connection picks out the horizontal vectors). We now verify that an auto-parallel frame field $e^{\parallel}$ on a curve $\gamma(t)$ on M induces a horizontal curve $\hat{\gamma}(t)$ on LM i.e $< \gamma, \dot{\hat{\gamma}} >= 0$.

Let $\hat{\gamma}(t)$ be represented by $x^i(t), y^a_b(t)$ then $e^{\parallel}_a = y^b_a e_b$

$$
\begin{aligned}
0 &= \nabla_{\dot\gamma} e_a^{\|} = \nabla_{\dot\gamma}(y_a^b e_b) \\
&= (\nabla_V y_a^b) e_b + y_a^b \nabla_V e_b \\
&= v^i \frac{\partial}{\partial x_i} y_a^b e_b + y_a^b < \hat\omega_a^c, \dot\gamma > e_c \\
&= \dot y_a^b e_b + y_a^b < \hat\omega_a^c, \dot\gamma > e_c \text{ for } \dot\gamma = \dot x^i \partial_i \\
&= y_a^b e_b + y_a^b < \hat\omega_a^b, \dot x^i \partial_i > e_b
\end{aligned}
$$

All this implies that $-y_a^b < \hat\omega_a^b, \dot x^i \partial_i > e_b = \dot y_a^b e_b$. This means that $\hat{\dot\gamma} \equiv \dot x^i \partial_i + \dot y_b^a \partial_a^b = \dot x^i \partial_i - y_a^b < \hat\omega_a^b, \dot x^i \partial_i > \partial_a^b = \dot x^i \left(\partial_i - y_a^b < \hat\omega_a^b, \partial_i > \partial_a^b\right)$. This is the expression we had for a general horizontal vector in (6.5).

3 Principal Bundles

2 manifolds P and M are to be given along with a smooth surjective map $\pi : p \to M$. All pre-images being submanifolds of P diffeomorphic to each other. Aright vertical action of a lie group G in the total space P is to be added. i.e

$$R_g : P \to P, \quad R_{gh} = R_h \circ R_g, \quad \pi \circ R_g = \pi$$

The action is free (all stabilizers are trivial) and transitive(any 3 pts in a single fiber can be joined by the action). There is a local product structure which may or may not turn out to be global

$$
\begin{aligned}
\psi_\alpha : \pi^{-1}(\mathcal{O}_\alpha) \to \mathcal{O}_\alpha \times G \\
\psi_\alpha : p \to (m, h) \implies pg \to (m, hg)
\end{aligned}
$$

For any 2 points p, p' residing in a common fiber $\pi(p) = \pi(p')$ $\exists!$ group element $g \in G$ which links the points in the sense that $p' = pg$ Why? The action of the group is free and transitive. Let G be a lie group, H a closed lie subgroup . Each homogeneous space $M\ G/H$ is the base of a principal H-bundle $\pi : G \to M\ G/H$. We can check in fact we have a principal bundle

a) Any two domains $\mathcal{O}_\alpha, \mathcal{O}_\beta$ that lie in a coset of G/H will be taken to the fiber in the total space, so that $\pi^{-1}(\mathcal{O}_\alpha) \simeq \pi^{-1}(\mathcal{O})_\beta$
b)For the right vertical action of the lie group in the total space we have $g \in G, g \longmapsto [g], \tilde g \in H\ g\tilde g \longmapsto [g\tilde g] = [g] \implies \pi \circ R_{\tilde g} = \pi$
c) Local product structure

$$
\begin{aligned}
\psi_\alpha = \pi^{-1}(\mathcal{O}_\alpha) \to \mathcal{O}_\alpha \times G \\
\psi_\alpha : g \longmapsto ([g], h) \implies g\tilde g \to ([g], g\tilde g)
\end{aligned}
$$

There is also additional structure, namely a fiber preserving $\mathcal{L}_g$ (left action) of the group G on the total space, as well as its projection L_g being left action of G on the base M i.e $\pi \circ \mathcal{L}_g = L_g \circ \pi$.

The generators of the action H along fibers are the left invariant field L_Y on G corresponding to elements Y which belong to the sub-algebra $\Xi \subset \mathcal{G}$ Why?

Theorem 3.1 *The flow of left-invariant vector fields is generated by right translation. The flow of right invariant vector fields is generated by left translations.*

Proof LetL_X be a left invariant vector field and $\Phi_t^{L_X}$ the flow of left invariant vector fields. But we know that $L_{g*}L_X = L_X$ and the flow of $L_{g*}L_X$ is $L_g \circ \Phi \circ L_g^{-1}$, which is also the flow of $L_X \implies L_g \circ \Phi_t^{L_X} \circ L_{g^{-1}} = \Phi_t^{L_X} \implies L_g \circ \Phi_t^{L_X} = \Phi^{L_X} \circ L_g$. So the flow commutes with all left translation but we know that $L_g \circ R_h = R_h \circ L_g$ so $\Phi_t^{L_X}$ is got by a right translation. The similar argument can be made for a right invariant vector field $\Box$

The generators of the $\mathcal{L}_g$ action in the total space G are all right-invariant fields R_x G with $X \in \mathcal{G}$ i.e

$$\frac{d}{dt}\left(\mathcal{L}_{exp(tX)}g\right) = \hat{\xi}_X \qquad \text{where } \hat{\xi}_X \text{ is a right invariant vector field} \tag{6}$$

Examples

1. $SL(2,\mathbb{C})$ *bundle*

$G = SL(2,\mathbb{C}), H =$ stabilizer of $\begin{pmatrix}1\\0\end{pmatrix}$. The stabilizer matrix is of general form $\left(\begin{smallmatrix}1 & b\\ 1 & a\end{smallmatrix}\right)$. But these must have determinant 1 this means that $a = 1$ and lower left hand corner is actually zero so that we have an element of H looking like $\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right)$. This is isomorphic to $\mathcal{C}$. What about the orbit of $\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)$? This is $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}a & \\ c & \end{smallmatrix}\right)$ Which isomorphic to $\mathbb{C}^2 \equiv M \simeq G/H$ It turns out that this H-bundle is trivial. To show this we provide a global section. $\psi : \left(\begin{smallmatrix}a & \\ c & \end{smallmatrix}\right) \to \begin{pmatrix}a & \frac{\bar{c}}{k}\\ \bar{c} & \frac{\bar{a}}{k}\end{pmatrix}$ where $|a|^2 + |c|^2 = k$. From all this we can now see that $SL(2,\mathbb{C})$ is diffeomorprhic to $\mathbb{R}^3 \times \mathbb{S}^3$ because $SL(2,\mathbb{C}) \simeq M \times H = \mathbb{C}^2 \times \mathbb{C} \simeq R \times S^3 \times (\mathbb{C} \simeq \mathbb{R}^2) \simeq \mathbb{R}^3 \times \mathbb{S}^3$

2. Proper orthochronous Lorentz group. $G = L_+^\uparrow$, H = stabilizer of $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

a) the orbit of the point is upper hyperboloid.
b)as a manifold , $M \simeq \mathbb{R}^3$
c) the group H is isomorphic to $SO(3)$

Note: If the base manifold is contractible then the bundle is globally trivial. Since $\mathbb{R}^3$ is contractible then $L_+^\uparrow = \mathbb{R}^3 \times SO(3)$

3. Hopf Bundle

Consider $\mathbb{C}^2$ with elements χ and $\mathbb{R}^3$ with elements $x \to x_a$ $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. $SU(2)$ acts on both manifolds as $\chi \longmapsto A\chi \equiv L_A\chi, \forall A \in SU(2)$ and on $\mathbb{R}^3$ through 2-sheeted covering $f : SU(2) \longmapsto SO(3)$ so that $(f(A)x) = \hat{L}_{f(A)}x$ and

We first need to understand why $SU(2) \simeq \mathbb{S}^3$ to construct the hopf bundle. The construction comes from quaternions$\{I, \mathbf{i} = i\sigma_z, \mathbf{j} = i\sigma_y, \mathbf{k} = i\sigma_x\}$ hence a general quaternion is $A = aI + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix}$ where $x = a + ib, y = c + id$

The inner-product of quaternions is $(x_1, y_2) \cdot (x_2, y_2) = x_1\hat{x} + y_1\hat{y_2}$. So if we have unit quaternions i.e $x_1\hat{x} + y_1\hat{y_2} = 1$ then $A \in SU(2)$ but because $detA = a^2 + b^2 + c^2 + d^2 = 1$ we have that $SU(2) \simeq \mathbb{S}^3$.

Going back the construction of a hopf bundle we define a *non-linear* map

$$\pi : \mathbb{C}^2 \to \mathbb{R}^3 \qquad \chi \longmapsto \mathbf{r} \qquad \mathbf{r} : \chi^\dagger \sigma \chi \tag{7}$$

Since the matrices $\{I, \sigma_x, \sigma_y, \sigma_z\}$ span all hermitian matrices the matrix $\chi\chi^\dagger$ may be parametrized as $\chi\chi^\dagger = \frac{1}{2}(rI_2 + \mathbf{r} \cdot \sigma)$ where $r := \chi^\dagger\chi$. We can now restrict π to $\mathbb{S}^3 \subset \mathbb{C}^2$ of radius $\sqrt{r}$ in $\mathbb{C}^2$ so to columns of χ which satisfy $\chi^\dagger\chi = r$. This restriction has a two dimensional sphere at an image. To see that one calculates $|\mathbf{r}|^2 = (\chi^\dagger\sigma)(\chi\chi^\dagger)\sigma\chi) = r^2$. We can normalizer r so $r^2 = 1$. In this case we get a restricted map $\tilde{\pi} : \mathbb{S}^3 \to \mathbb{S}^2$ with $\eta \longmapsto \mathbf{n}$ with

$$\eta\eta^\dagger = \frac{1}{2}(I_2 + \mathbf{n} \cdot \sigma) \text{ where } \mathbf{n} := \eta^\dagger \sigma \eta \tag{8}$$

The map $\tilde{\pi}$ is $SU(2)$ equivariant i.e

$$\tilde{\pi} \circ L_A = \hat{L}_{f(A)} \circ \tilde{\pi} \tag{9}$$

this means that the following actions are equivalent $\eta \longmapsto A\eta$ or $\mathbf{n} \longmapsto R\mathbf{n}$ where $A \in SU(2), R \in SO(3)$. Plus the map $\tilde{\pi}$ is surjective because each vector $\mathbf{n}$ has a pre-image since $\mathbf{n} \to \eta\eta^\dagger = n_i\sigma^i$. Next we note that if η is the pre-image of $\mathbf{n}$ then so is $e^{i\alpha}\eta$ since $e^{i\alpha}\eta\eta^\dagger e^{-i\alpha} = \eta\eta^\dagger$. So really the what we have is the following

$$e^{i\alpha}\eta \longmapsto \mathbf{n} \tag{10}$$

We now think of $\tilde{\pi}$ as a projection map on some bundle. Looking at (6.10) we see that there is a fiber at a point on $\mathbb{S}^2$ being $U(1) \simeq \mathbb{S}^1$.Therefore locally

$\mathbb{S}^3 \simeq \mathbb{S}^2 \times \mathbb{S}^1$ and the fiber bundle $\tilde{\pi} : \mathbb{S}^3 \to \mathbb{S}^2$ is called the *hopf bundle*

4 The Connection and parallel transport on a Principal bundle

4.1 The connection

Definition A *connection* on a principal G-bundle $\pi : P \to M$ is an arbitrary horizontal G-invariant distribution $\mathcal{D}^h$ on total space P

All horizontal subspaces within a single fiber may be linked together by the action of the group G. Same is true for vertical subspaces i.e $R_{g*}Hor_pP = Hor_{pg}P$ and $R_{g*}Ver_pP = Ver_{pg}P$.

The decomposition of a vector into Hor and Ver commutes with the group i.e $horR_{g*}V = R_{g*}horV$.
Given $X \in \mathcal{G}$ consider the fundamental field ξ_X of the action R_g on P. Define a map

$$\Psi_p : \mathcal{G} \to Ver_pP \qquad X \to \xi_X(p) \tag{11}$$

Ψ_p is an isomorphism since Ψ_p is a representation of the lie algebra $\mathcal{G}$, plus if $\xi_X(p) = \xi_Y(p)$ then $\xi_{X-Y}(p) = 0$ and $X - Y$ is in the stabilizer which is trivial so $X = Y$. Therefore we have injectivity and the action is transitive so we have surjectivity.

We also have the following behavior

$$R_{g*} \circ (\xi_X) = \xi_{Adg^{-1}X}(pg) = R_{g*} \circ \Psi_p$$
$$\Psi_{pg} \circ Ad_{g^{-1}} = \Psi_{pg} \circ (g^{-1}Xg) = \xi_{Ad_{g^{-1}}X}(pg) \implies R_{g*} \circ \Psi_p = \Psi_{pg} \circ Ad_{g^{-1}}$$

Aslo we have $\Psi_{pg}^{-1} \circ R_{g*} = Ad_{g^{-1}} \circ \Psi_p^{-1}$.

Now we introduce the connection form. At a point p $\in P$ define a lie algebra -valued 1-form by the prescription

$$< \omega_p, v_p > := \Psi_p^{-1}(Ver_p) \tag{12}$$

Clearly the horizontal vectors are annihilated by the 1-form since $\Psi_p^{-1}(horP) = 0$.

If E_i is an arbitrary basis of the lie algebra then $\omega_p = \omega_p^i E_i$. Where we can think of ω_p^i as 1-forms and serve as the constraint 1-forms of the horizontal subspace $Hor_pP \subset T_pP$

4.2 Parallel Transport and the Exterior co-variant derivative

Let $\mathcal{D}^h$ be a horizontal distribution on P, given by connection form ω. We can check that if $v \in T_xM$ is an arbitrary vector at a point $x \in M$, then at each point $p \in \pi^{-1}(x)$ in the fiber over x $\exists$ unique horizontal lift such that $\pi_* v^h = v, v^h \in Hor_pP$ and $v^i\partial_i \longmapsto v^iH_i$ which is clearly an isomorphism.

Also $\pi_* : T_pP \to T_xM$ gives a decomposition $T_pM = Ver_pP \bigoplus Hor_pP$ where $Ver_pP := ker\pi_* \implies Hor_pP \simeq T_xM$.

If we lift v to v^h to all points of the fiver over x, we get a vector field which is G-invariant . $R_{g*}v^h = v^h$ again $v^i\partial_i \to v^iH_i$ and $< \omega, v^h >= 0 = \omega(v^h)$ and $\omega(R_{g*}v^h) = R_g^*\omega(v^h) = 0 \implies R_{g*}v^h \in Hor_{pg}P$.

Let γ be a curve on M of principal bundle $\pi : P \to M$ with connection and p an arbitrary point from the fiber over $\gamma(0)$. There exists a unique curve γ^h

on P given by conditions

$$\pi \circ \gamma^h = \gamma$$
$$\gamma^h(0) = p$$
$$< \omega, (\dot{\gamma}^h) >= 0$$

γ can be made into an integral curve of some vector field V. It will be unique because of uniquenes and existence of solutions of ODE. The vector field can then be lifted and since there is an isomorphism betweenT_xM and Hor_pP there will be a unique lift on the principal bundle with a corresponding unique integral curve for the lift. This will be γ^h.This procedure makes the claim obvious. All this means is that $\dot{\gamma}^h = (\dot{\gamma})^h$ and $(\gamma \circ \sigma)^h = \gamma^h \circ \sigma, \sigma : t \to \sigma(t) \in \mathbb{R}$ where σ is a re-parametrization.

We interpret points of the horizontal lift $p(t) \equiv \gamma^h(t)$ as a parallel transported "generalized frame". We can transport more general objects. These are equivariant functions on P i.e maps

$$\Phi : P \to (v, \rho) \tag{13}$$
$$\Phi \circ R_g = \rho(g^{-1}) \circ \Phi \text{ i.e } \Phi(pg) = \rho(g^{-1})\Phi(p) \tag{14}$$

A quantity of type rho at a point $x \in M$ is introduced as an equivariant function Φ from the fiber over x to V. Its value $Phi(p)$ is regarded as "components" of the quantity with respect to "basis" p. If we change "basis" i.e $p \to pg$ then $\Phi(p) \to \Phi(pg) \equiv \rho(g^{-1})\Phi(p)$

Parallel transport of Φ can be defined in the same we defined parallel transport of frames i.e if we parallel transport a frame along a curve γ, the corresponding components are kept constant (by definition). This means

$$\Phi \text{ is parallel transported along } \gamma \implies \Phi(\gamma^h) = \text{ constant}$$

The above condition may be re-stated as $< d\Phi, (\dot{\gamma})^h >$. An explicit calculation can confirm this

$$< d\Phi, (\dot{\gamma})^h >= d\Phi(\dot{\gamma}^h) = \Phi_* = \frac{d}{dt}\left(\Phi(\gamma^h)\right)|_{t=0} = \frac{d}{dt}(\text{ constant}) = 0 \tag{15}$$

The fact that Φ corresponds to an autoparallel quantity of type ρ on a curve γ may be written in terms of forms, namely a projection on the horizontal part and a combination of such a projection with exterior derivative. For an arbitrary p-form α on P, define a new p-form $hor\alpha$ by

$$(hor\alpha) = (U, \dots V) := \alpha(horU, \dots horV) \tag{16}$$

i) it is well-defined (result is a p-form) since $\Gamma : V \longmapsto horV$ is a linear operation and an isomorphism.

ii) the map $hor : \Omega^p(P) \to \Omega^p(P)$ is a projection since $hor \circ hor = hor$. We check this by note $(hor \circ hor)\alpha(U, \dots V) = (hor\alpha)(horV, \dots horV) = \alpha(hor \circ horU, \dots hor \circ horV) = \alpha(horU, \dots horV)$

iii) horizontal forms are annihilated by a vertical argument and $hor\alpha = \alpha \iff i_W\alpha = 0$ for $W \in VerP$. On direction we have $hor\alpha(W, \dots) = \alpha(horW, \dots) = \alpha(W, \dots) = 0$ since $hor\alpha = \alpha \implies i_W\alpha = 0$. In the other direction we have $i_W\alpha = 0 = \alpha(W, \dots) = 0, \alpha(horW, \dots) = (hor\alpha)(W, \dots) \implies \alpha(horW, \dots) = \alpha(W, \dots) = 0 \implies hor\alpha = \alpha$

iv) For connection we have $hor\omega = 0$ since $\omega(V) := \Psi^{-1} \circ verV, v \in T_pP \implies hor\omega = \omega(horV) = \Psi^{-1}(ver \circ horV) = 0$

v) $hor(\alpha + \lambda\beta) = hor\alpha + \lambda hor\beta$ and $hor(\alpha \wedge \beta) = hor\alpha \wedge hor\beta$. This means that hor is an endo-morphism of the cartan algebra $\Omega(P)$ of differential forms

$$\bar{\Omega}(P) := Im\ hor \subset \Omega(P)$$

The map hor preserves the representation of α. WE can now define the exterior covariant derivative by

$$\mathcal{D}\alpha := hor\ d\alpha \tag{17}$$

i) it is map $\mathcal{D} : \Omega^p(P) \to \bar{\Omega}^{p+1}(P)$

ii) it behaves like $\mathcal{D}(\alpha+\lambda\beta) = \mathcal{D}\alpha+\lambda\beta, \mathcal{D}(\alpha\wedge\beta) = (\mathcal{D}\alpha)\wedge hor\beta+(\hat{\eta}\ hor\alpha)\wedge\beta$

iii) $\mathcal{D}$ preserves the representation of α since hor does.

Suppose a quantity Φ of type ρ satisfies the condition $\mathcal{D}\Phi = 0$ over some domain U on M. If γ is any curve passing in U, then the restriction of Φ to fibers over γ corresponds to an auto-parallel quantity of type ρ defined over γ since $< d\Phi, (\dot{\gamma})^h >=< d\Phi, hor(\dot{\gamma})^h >=< hord\Phi, (\dot{\gamma})^h >=< \mathcal{D}\Phi, (\dot{\gamma})^h >$

4.3 Curvature form and explicit expression for exterior-covariant derivative

The lift of curve γ, γ^h, is not uniquely determined i.e two paths starting at x on M may end at different points in the fiber once the horizontal lift is performed. We are only guaranteed to land in the same fiber. Another way of staring this is parallel transport around a loop may be non-trivial, whether this is true or not depends on the integrability of the horizontal distribution. If the horizontal distribution is integrable, then the lift of a small enough lop lies entirely within the integral sub-manifold passing through the point x. But since there is only one integrable sub-manifold along x the end and starting points necessarily coincide. Therefore integrability determines if parallel transport is locally dependent. So we need a way of measuring non-integrability.

Frobenius integrability condition of a distribution states that the distribution is integrable $\iff$ the restriction of the exterior derivative of all constraint 1 forms to $\mathcal{D}$ vanishes i.e $d\theta^i = 0$ on $\mathcal{D}$ or $\mathcal{D}^h$ is integrable $\iff \{U, V \in \mathcal{D}^h \implies d\omega(U, V) = 0\}$.

So the horizontal distribution $\mathcal{D}^h$ is integrable $\iff$ the 2-form $\Omega \equiv hord\omega$ vanishes i.e

$$\mathcal{D}^h \text{ is integrable } \iff \Omega := \mathcal{D}\omega = 0 \tag{18}$$

This 2-form is called the curvature form. If $\mathcal{D}^h$ is integrable $\implies \mathcal{D}\omega = hord\omega = d\omega(horU, horV) = 0$ and $\impliedby$ But if $d\omega(horU, horV) = 0 \implies \mathcal{D}^h$ is integrable.

This condition is simple but we still lack a simple algorithm for computing the curvature form. We now develop that.

Let α and β be differential forms with values in a lie algebra $\mathcal{G}$ then

i) $\alpha = \alpha^i E_i$, $\beta = \beta^i E_i$ then the prescription

$$[\alpha \wedge \beta] := \alpha^i \wedge \beta^j [E_i, E_j] \equiv c_{ij}^k \alpha^i \wedge \beta^j E_k \tag{19}$$

is well defined i.e it does not depend on the basis of the lie algebra. This is true since if A changes the basis then A^{-1} changes the components i.e α^i, β^j and the net effect is nothing.

ii) $[\alpha \wedge \beta] = -(-1)^{pq}[\beta \wedge \alpha]$. So there is a negative sign got by changing order of differential forms and another changing order of elements in the lie bracket.

iii) $[\alpha \wedge \beta](U, V) := [\alpha(U), \beta(V)] + [\beta(U), \alpha(V)]$ so a corollary is $[\alpha \wedge \alpha](U, V) = 2[\alpha(U), \alpha(V)]$. Note that extra sign coming from switching lie algebra elements in the commutator prevents us from getting zero.

We can use definition of wedge product to get the results

iv) $d[\alpha \wedge \beta] = [d\alpha \wedge \beta] + [\hat{\eta}\alpha \wedge d\beta]$ where $\eta = (-1)^p$

v) $hor[\alpha \wedge \beta] = [hor\alpha \wedge hor\beta]$

So the curvature form $\Omega \equiv = \omega$ may be expressed as $\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$. To prove this we have to prove that $d\omega(horU, horV) = d\omega(U, V) + [\omega(U), \omega(V)]$. We need to prove this for an arbitrary basis . So let us say that $U_p = \hat{U}_p + \xi_x$ where $\hat{U}_p \in HorP$ and $\xi_x \in VerP$. If U and V are entirely horizontal then it is easy to see that the claim is true. The two slight more non-trivial cases are if one is entirely horizontal and the other entirely vertical and lastly if both are entirely vertical. In either case the hard part is the first term on the right hand side.

$$d\omega(\xi_x, \hat{U}) = i_{\hat{U}} i_{\xi_x} d\omega = 0$$
$$d\omega(\xi_x, \xi_y) = i_{\xi_y} i_{\xi_y} d\omega = -[X, Y]$$

Using the above results the claim follows. Both results come from the identity $i_{\xi_x} d\omega = -ad_X\omega = -[X, \omega] \equiv -c_{ij}^k X^i \omega^j E_k$. Let us prove this

Recall $R_g^*\omega = Ad_{g^{-1}}\omega \implies \frac{d}{dt}(R_g^*\omega)|_{t=0} = \frac{d}{dt}(Ad_{g^{-1}}\omega)$ Now $\mathcal{L}_V = \frac{d}{dt}\Phi^*|_{t=0}$ where Φ^* is the flow of V. Right translations are the flow of left invariant vector fields so $\frac{d}{dt}(R_g^*\omega) = \mathcal{L}_{\xi_x}(\omega)$. The Ad map has the lie algebra version as ad $\implies \mathcal{L}_{\xi_x}\omega = -ad_X\omega \equiv= -[X, \omega]$. We can calculate $i_{\xi_x} d\omega$ as follows $i_{\xi_x} d\omega = (\mathcal{L}_{\xi_x} - di_{\xi_x})\,\omega = \mathcal{L}_{\xi_x}\omega - dX = -[X, \omega]$
Amazing consequence: The operation *hor* is realized as simple addition of the term $\frac{1}{2}[\omega \wedge \omega]$. We thus do not bother with horizontal directions.
The curvature form has the following properties

i) $hor\Omega = \Omega$ because we showed $hor \circ hor = hor$

ii) $R_g^*\Omega = Ad_{g^{-1}}\Omega$ because $R_g^* \circ hor = hor \circ R^*g$ so $R_g^*\Omega = R_g^* hor d\omega = hor R_g^* d\omega = hor d R_g^*\omega = hor d Ad_{g^{-1}}\omega = Ad_{g^{-1}}\Omega 0$

iii) Ω satisfies Bianchi identity $\mathcal{D}\Omega = \mathcal{D}\mathcal{D}\omega = 0$ because $\mathcal{D}\mathcal{D}\omega = hor d(d\omega + \frac{1}{2}[\omega \wedge \omega]) = hor[d\omega \wedge \omega] = [\mathcal{D}\omega \wedge hor\omega] = [\mathcal{D}\omega \wedge 0] = 0$.

On the total space P, consider a differential p-form α with values in the lie algebra $\mathcal{G}$ and a q-from with values in a representation space $(W, \rho\ )$ of the lie algebra then $\alpha = \alpha^i E_i$ and $\beta = \beta^a E_a$ (E_a basis for W). We then define the following operations

$$\rho(\alpha)\dot{\wedge}\beta := \alpha^i \wedge \beta^a \rho(E_i) E_a \tag{20}$$
$$= \rho_{bi}^a(\alpha^i \wedge \beta^b) E_a \tag{21}$$
$$= (\alpha_b^a \wedge \beta^b) E_a \tag{22}$$

The above exterior product is independent of choice of basis and therefore is well defined. If we have a 0-form Φ we have

$$\rho(\alpha)\dot{\wedge}\Phi = \rho(\alpha) \wedge \Phi$$
$$= \alpha^i \rho(E_i) \Phi^a E_a$$
$$= (\rho_{bi}^a \alpha^i \Phi^b) E_a \qquad = (\alpha_b^a \Phi^b) E_a$$

There other exterior product defined in (6.19) is a special case of (6.20) where rho is simply the adjoint representation. So $\rho(\alpha) = ad\alpha = [\alpha^i E_i,]$. In the following we state a few properties of this exterior derivative

i) $d(\alpha\dot{\wedge}\beta) = d(\alpha^i \wedge \beta^a \rho(E_i)E_a) = d\alpha^i \wedge \beta^j \rho(E_i)E_a + \hat{\eta}\alpha^i \wedge d\beta^j \rho(E_i)E_a$ so $\rho(d\alpha)\dot{\wedge}\beta + \hat{\eta}\rho(\alpha)\dot{\wedge}d\beta$

ii) $hor\alpha\dot{\wedge}\beta = hor\alpha\dot{\wedge}hor\beta$

If α is of type Ad and β is of type ρ then $\alpha\dot{\wedge}\beta$ is of type ρ.

$$\begin{aligned} R_g^*(\rho(\alpha\dot{\wedge}\beta)) &= \rho(R_g^*\alpha)\dot{\wedge}R_g^*\beta \\ &= \rho(Ad_{g^{-1}}\alpha)\dot{\wedge}\rho(g^{-1})\beta \\ &= \rho(g^{-1}\alpha g) \wedge \rho(g^{-1})\beta \\ &= \rho(g^{-1})[\rho(\alpha) \wedge \beta] \end{aligned}$$

Let $\pi : P \to M$ be a principal G-bundle , $\omega = \omega^i E_i$ a connection form, $\alpha = \alpha^a E_a$ a horizontal p-form of type ρ, $\Phi = \Phi^a E_a$ a function of type ρ and denote $\omega_b^a = \rho_{bi}^a \omega^i$ then

$$\mathcal{D}\alpha = d\alpha + \rho(\omega)\dot{\wedge}\alpha \tag{23}$$
$$\mathcal{D}\Phi = d\Phi + \rho(\omega)\Phi \tag{24}$$

Why? Decompose the vector fields into horizontal parts U^h and vertical parts U^v

If all the vector fields in (6.23) are horizontal then the identity trivially holds. If all the vector fields are horizontal except for a few where "a few" means more than one, then one of the horizontal vector fields will be eaten by α which is horizontal. If all the vector fields are vertical then we get 0=0 and identity holds. The hard case is if we have one vertical field and the rest are horizontal.

$$hord\alpha(U_1^h, U_2^h, \dots U_n^v) = i_{U_n^v}\alpha(U_1^h, \dots U_{N-1}^h) + \rho(X)\alpha(U_1^h, \dots, U_{N-1}^h) \tag{25}$$

Now note that $\frac{d}{dt}(R_g^*\alpha)|_{t=0} = \mathcal{L}_{\xi_x}\alpha = i_{\xi_x}dd\alpha + di_{\xi_x}\alpha = i_{\xi_x}d\alpha$ but also $R_g^*\alpha = \rho(g^{-1})\alpha \implies \frac{d}{dt}(R_g^*\alpha)|_{t=0} = \frac{d}{dt}(\rho(g^{-1})\alpha)|_{t=0}$ and therefore $i_{\xi_x}d\alpha = -\rho(X)\alpha$. Using this result in (6.25) the result the identity follows. For (6.24) if we simple have a horizontal field the result follows trivially. If the vector field is vertical U^v then $d\Phi(horU^v) = U^v(\Phi) + \rho(X)\Phi$. Now, is $U^v(\Phi) = -\rho(X)\Phi$?. The answer is yes since Φ is of ρ type.

4.4 Integrability of horizontal distributions and path independence of parallel transport

Earlier we saw that if a region is small enough so that it entirely lies within a specific integrable distribution then parallel transport was path independent. Now suppose the region is big. Consider 2 paths c_0 and c_1 such that both begin in x and terminate in y. Do their lifts lie in the same integrable manifold? If c_0 and c_1 are homotopic then this is true.If the paths fail to be homotopic the parallel transport may(does not need to) depend on path.

Complete parallelism: a linear connection for which there exists a convariantly constant frame field e_a such that $\nabla_V e_a = 0$. We have complete parallelism $\iff$ $\Omega = 0$. In terms of a connection form it gives $\omega^a_b = 0$. So $\Omega = d\omega^a_b + \omega^a_c \wedge \omega^c_d = 0$. Noe suppose $\Omega = 0$ does that mean there exists e_a in which $\omega^a_b = 0$? IT turns out $\Omega = 0 \implies \exists\sigma : \sigma^*\omega = 0$ where σ is a section. $\Omega = 0$ says that the horizontal distribution is integrable. Now let S be an integral sub-manifold which is the image of section σ from some domain on M. This is a horizontal section i.e a section such that all vectors tangent to it are horizontal. Now pull back ω to the base manifold like $\hat{\omega} = \sigma^*\omega$ now $< \sigma^*\omega, v >=< \omega, \sigma_* v >= 0$ since $\sigma_* v$ is horizontal.