1 Tangent and Co-Tangent Bundle

Let M be a smooth manifold and let T_xM be a the tangent space at a point $x \in M$. Define the set TM as the collection(disjoint) of all tangent spaces at all points of M.

$$TM := \bigcup_{x \in M} T_xM \tag{1}$$

Define a surjective map called the canonical projection

$$\pi : TM \to M \qquad v \longmapsto x \tag{2}$$

We can endow this collection with a smooth structure.Let x^a be a local co-ordinate in neighborhood $\mathcal{O}$ of a point x i.e let $\Psi : \mathcal{O} \to \mathbb{R}^n[x^1, x^2, \dots, x^n]$ be a chart. Consider the domain $\hat{\mathcal{O}} : \pi^{-1}(\mathcal{O}) \subset TM$. Then we introduce on $\hat{\mathcal{O}}$ canonical co-orindates as follows: if $u \in \hat{\mathcal{O}} \implies v \in TM$ for some $x \in \mathcal{O}$ then $v = v^a \frac{\partial}{\partial x^a}|_x$, $(v^1, V^2, \dots v^n) \in \mathbb{R}^n$.

So the 2n- tuple of numbers $\left(x^1, \dots, x^n, v^1, \dots v^n\right)$ uniquely corresponds to a point $v \in \hat{\mathcal{O}}$ and so $\hat{\Psi} : \hat{\mathcal{O}} \to \mathbb{R}^{2n}[x^1, \dots x^n, v^1, \dots v^n]$. $\hat{\mathcal{O}}$ is a chart on TM. If there is a change of co-ordinates $x^a \longmapsto x^{'a}$ then $(x^a, v^a) \longmapsto (x^{'a}, J^a_b(x)v^b)$ where $J^a_b = \frac{\partial x^a}{\partial x^b}$. Moreover $\{\hat{\mathcal{O}}, \hat{\Psi}\}$ is a smooth atlas because in the intersection of two charts we can apply the Jacobian transformation which will involve smooth relations. Plus TM is always an orientable manifold because the jacobian of a co-ordinate change is always positive

$$\hat{J} = \frac{\partial(x', v')}{\partial(x, v)} = J^2 > 0$$

Another important manifold is denoted by T^*M. It is a set of all cotangent spaces at all points of M.

$$T^*M := \bigcup_{x \in M} T^*M \tag{3}$$

There is a corresponding canonical projection map denoted by letter τ

$$\tau : T^*M \to M \tag{4}$$

We can introduce a smooth structre as well. If $p \in T * M$ its decomposition with respect to a co-ordinate basis is

$$p = p_a dx^a \qquad (p_1, \dots p_n) \in \mathbb{R}^n$$

so $\left(x^1, \dots x^n, p_1, \dots p_n\right)$ uniquely corresponds to a point $p \in \hat{\mathcal{O}}$, $\tau^{-1}(\mathcal{O})$ so that $\hat{\Psi} : \hat{\mathcal{O}} \to \mathbb{R}^{2n}[x^1, \dots, x^n, p_1, \dots p_n]$ is a chart on $\hat{\mathcal{O}} \subset T * M$. If there is a change of co-ordinates then $(x^a, p_a) \longmapsto \left(x^{'a}, (J^{-1})^b_a(x)p_b\right)$

Let M be a part of the plane $\mathbb{R}^2$ in which both cartesian co-ordinates (x, y) and polar co-ordinates operator. Then on TM we have the following relations

$$
\begin{aligned}
x &= r\cos\phi \\
y &= r\sin\phi \\
r &= \sqrt{x^2+y^2} \\
\phi &= \tan^{-1}(\frac{y}{x})
\end{aligned}
$$

also we have $p_x dx + p_y dy = p_r dr + p_\phi d\phi$ and $J^{-1} = \frac{\partial x^b}{\partial x'^a}$. So we have the following calculation

$$
\begin{aligned}
p_r &= \frac{\partial}{\partial r}(r\cos\phi)p_x + \frac{\partial}{\partial r}(r\sin\phi)p_y = \cos\phi p_x + \sin\phi p_y \\
p_\phi &= \frac{\partial}{\partial \phi}(r\cos\phi)p_x + \frac{\partial}{\partial \phi}(r\sin\phi)p_y = -r\sin\phi p_x + r\cos\phi p_y
\end{aligned}
$$

and so

$$
\begin{aligned}
p_r &= \cos\phi p_x + \sin\phi p_y \\
p_\phi &= -r\sin\phi p_x + r\cos\phi p_y
\end{aligned}
$$

A similar calculation can be done to show that

$$
\begin{aligned}
p_x &= p_r\cos\phi - p_\phi\frac{\sin\phi}{r} \\
p_y &= p_r\sin\phi + p_\phi\frac{\cos\phi}{r}
\end{aligned}
$$

2 General Concept of a Fiber Bundle

A generalization of the concepts discussed is if we pasted at each point $x \in M$ a general manifold F_x all of which are diffeomorphic to F (i.e if $x, x' \in M$ then $F_x \simeq F_{x'} \simeq F$). The manifold F is called a typical fiber, F_x is the fiber over a point x, M is the base and

$$
\mathcal{B} := \bigcup_{x\in M} F_x \qquad \text{is the total space} \tag{5}
$$

All these elements when taken together constitute a structure called the Fiber Bundle. It has a canonical projection map

$$
\pi : \mathcal{B} \to M \tag{6}
$$

All the preimages $F_x \equiv \pi^{-1}(x)$ are required to be diffeomorphic to a common manifold F. Lastly, there is a requirement of a local product structure: $\exists$ a covering $\mathcal{O}_a$ of the base M and a system of

$$\Psi_a : \pi^{-1}(\mathcal{O}_a) \to \mathcal{O}_a \times F$$

Ψ_a is called a local trivialization such that $\pi_1 \circ \Psi_a = \pi$ where $\pi_1 : M \times F \to M$.

Fiber bundles can be mapped to one another

$$\pi : \mathcal{B} \to M \rightarrowtail \pi' : \mathcal{B}' \to M'$$

The bundle map is a pair of maps $f, \hat{f}$ defined so that the following diagram commutes

$$\begin{tikzcd} \mathcal{B} \arrow[r, "f"] \arrow[d, "\pi"'] & \mathcal{B}' \arrow[d, "\pi'"'] \\ M \arrow[r, "\hat{f}"'] & M' \end{tikzcd}$$

A trivial bundle is equivalent to a product bundle through a diffeomorphism

$$f : \mathcal{B} \to M \times F$$

which obeys $\pi_1 \circ f = \pi$ and f is called a global trivialization. For sufficiently small pieces (charts) any bundle is trivial. But the pieces are glued together in such a way that a resulting bundle need not be globally trivial.

Examples.
1. Cylinder $\to$ trivial bundle
$S^1 \times \mathbb{R} \to S^1$

2. Mobius Band not trivial
Glued together so that for any 2 charts $(x, y), (x', y')$ we have $x' = x + c, y = -y + c_2$ or $[0,1] \times [0,1]/\sim$ where $(0,t) \sim (1, 1-t)$

Another concept is that of a local section of a bundle $\pi : \mathcal{B} \to M$. It is a smooth map $\sigma : \mathcal{O} \to \mathcal{B}$, $\mathcal{O} \subset M$ such that $\pi \circ \sigma = id_{\mathcal{O}}$. σ maps a point $x \in \mathcal{O} \subset M$ to its own fiber since $\left(x^1, \ldots x^n\right) \longmapsto \left(x^1, \ldots x^n, v^1, \ldots v^n\right)$ and then $\left(x^1, \ldots x^n, v^1, \ldots v^n\right) \longmapsto \left(x^1, \ldots x^n\right)$. First map is σ and the second is the canonical projection. On a product bundle $\pi : M \times F \to M$ the section are in 1-1 correspondent with maps from $M \to F$ since each point $x \longmapsto F_x \sim F$. Bundles usually have additional structure like that of a linear space or homogeneous space. Tangent bundles and co tangent bundles have the structure of a vector space.

2.1 The maps T^*f and Tf

Let $f : M \to M$ be smooth map of manifolds and $\pi_M : TM \to M$, $\pi_N : TN \to N$ be corresponding tangent bundles. Remember $f_* = T_xM \to T_{f(x)}N$. Vectors on M may be however regarded as points on TM. Consequently, a further map $Tf : TM \to TN$ is induced given by prescription

$$(Tf)(v) := f_*v \tag{7}$$

The map Tf is therefore a collection of all f_* maps. The map Tf closes the commutating diagram belows

$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d, "\pi_M"'] & TN \arrow[d, "\pi_N"'] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

We also have the following composition formula

$$\begin{aligned} T(f \circ g) &= (f \circ g)_*(v) \\ &= (f \circ g)_*(v) \\ &= f_* \circ Tg \\ &= Tf \circ Tg \end{aligned}$$

Similarly for $T * M$ we can define $T^*f : T^*N \to T^*M$ with $f : M \to N$ is an injective map by $(T^*f)_\alpha = f^*\alpha$

Rememeber we pushforward vectors and pullback forms. The following diagram therefore commutes

$$\begin{tikzcd} T^*M \arrow[d, "\tau_M"'] & T^*N \arrow[l, "T^*f"'] \arrow[d, "\tau_N"'] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

We also have the following composition rule

$$\begin{aligned} T^*(f \circ g) &= T^*g \circ T^*f \\ &= (f \circ g)^*\alpha \\ &= (g^* \circ f^*)\alpha \\ &= T^*g \circ T^*f \end{aligned}$$

3 Vertical subspace, vertical vectors

Let $\pi : \mathcal{G} \to M$ be a fiber bundle. The existence of the projection π singles out in the tangent space of any point $b \in \mathcal{B}$ a vertical subspace.

$$Ver_b\mathcal{B} \leq T_b\mathcal{B} \qquad Ver_b := Ker\pi_* b \tag{8}$$

The most general vertical vector fields on TM and T^*M respectively are $V = v^i(x,v)\frac{\partial}{\partial v^i}$ and $W = W_i(x,p)\frac{\partial}{\partial p_i}$ where $(x^i, v^i), (x^i, p_i)$ are co-ordinates on the tangent bundle and the cotangent bundle respectively. The vertical subspaces on TM and T^*M span the vector $\left(\frac{\partial}{\partial v^1}, \dots \frac{\partial}{\partial v^n}\right), \left(\frac{\partial}{\partial_p 1}, \dots \frac{\partial}{\partial p_n}\right)$

3.1 Lifts on TM and T^*M

In the context of fiber bundles a <u>lift</u> is in general a procedure which assigns to a geometrical object on the manifold, a geometrical object on the total space $\mathcal{B}$ of the bundle $\pi : \mathcal{B} \to M$. We begin by lifting cureves from $M \to TM$. Let $\gamma : \mathbb{R} \to M, t \longmapsto \gamma(t)$ be a curve on M. Then the curve: $\hat{\gamma} : \mathbb{R} \to TM, t \longmapsto \dot{\gamma(t)}$ is called the natural lift of the curve γ from $M \to TM$. We can check that the lifted curve $\hat{\gamma}$ is always exactly "over" the curve γ i.e $\pi \circ \hat{\gamma} = \gamma$ as follows; the co-ordinate representation of $\dot{\gamma(t)}$ is $(\dot{x}_1(t), \dots, \dot{x}_n(t))$ so in the tangent bundle we get the point $(x_1, \dots, x_n, \dot{x}_1(t), \dots, \dot{x}_n(t))$. Once the projection map is applied we get $(x_1, \dots, x_n)$ which is a point on the curve.

Consider a vector $u \in M$, $u \in T_xM \equiv \pi^{-1}(x)$. We may associate a curve in the fiber $\pi^{-1}(x)$ over x with this vector $\sigma(t) = v + tu \ v \in \pi^{-1}(x)$. The tangent vector at $t = 0$ of the curve ins a vector at the point $v \in TM$.

$$u^{\uparrow} := \dot{\sigma(0)} \equiv \frac{d}{dt}|_{t=0}\,(v + tu) \tag{9}$$

This vector is called the <u>vertical lift</u> of the vector u to the point $v \in TM$. In co-ordinate representation the curve is $\sigma(t) = (x^1, x^2, \dots, x^n, v^1 + tu^1, v^2 + tu^2, \dots v^n + tu^n)$ and the resulting vertical lift has co-ordinates $(u^1, \dots, u^n)$ i.e $u^{\uparrow} \equiv \dot{\sigma(0)} \in Ver_vTM < T_vTM$ or explicitly $u^{\uparrow} = u^a\frac{\partial}{\partial v^a}$. A single vector u may be lifted in this way to each point in the fiber π^{-1} over x, giving rise to a vector field defined on the fiber. If $u = u^a\frac{\partial}{\partial x^a}$ on M the vertical lift (to each $\pi^{-1}(x) \forall x \in M$) generates a vector field on TM, which is called the vertical lift of the field u.

We can also generate a vector field on TM by a different procedure namely by considering the flow on M and applying the functor T to get a flow on TM

$$\begin{tikzcd}
TM \arrow[r, "T\Phi_t"] \arrow[d, "\pi_M"'] & TM \arrow[d, "\pi_M"'] \\
M \arrow[r, "\Phi_t"'] & M
\end{tikzcd}$$

$T\Phi_t$ i indeed a flow since $T\Phi_{t+s} = T(\Phi_t \circ \Phi_s) = T\Phi_t \circ \Phi_s$. This flow is generated by a vector field called the complete lift of the field V. Let Φ_t :

$M \to M$ generated by $V = V^a(x)\frac{\partial}{\partial x^a}$. The co-ordinate representation of the infinitesimal flow Φ_ϵ and $T\Phi_\epsilon$ read

$$\Phi_\epsilon : x^a \longmapsto x^a(\epsilon) = x^a + \epsilon V^a(x)$$
$$T\Phi_\epsilon : (x^a, v^a) \longmapsto (x^a(\epsilon), v^a(\epsilon)) = (x^a + \epsilon V^a(x), v^a + V^a_{,b} v^b)$$

The co-ordinate expression for the lifted field $\tilde{V}$ is $\tilde{V} = V^a \frac{\partial}{\partial x^a} + V^a_{,b}(x) v^b \frac{\partial}{\partial v^a}$. We can construct lifts on the cotangent bundle also. Consider a co-vector α at $x \in M$. We assign a curve in the fiber $\tau^{-1}(x)$

$$\sigma(t) := p + t\alpha \qquad p \in \tau^{-1}(x) \tag{10}$$

The tangent vector at $t = 0$ of the curve is a vector in the point $p \in T^*M$

$$\alpha^{\uparrow} := \dot{\sigma(0)} \equiv \frac{d}{dt}|_{t=0}(p + t\alpha) \qquad T_pT^*M \tag{11}$$

This is the vertical lift of the co-vector at $p \in T^*M$. the lift of a co-vector is a vector. The co-ordinate presentation of the curve $\sigma(t)$ is $x^a(t) = x^a, p_a(t) = p_a + t\alpha_a$ so

$$\alpha^{\uparrow} \equiv \dot{\sigma(0)} \in Ver_pT^*M \leq T_pT^*M \tag{12}$$

therefore $\alpha = \alpha_a dx^a$ gets lifted to $\alpha^{\uparrow} = \alpha_a \frac{\partial}{\partial p_a}$. The procedure of constructing a complete lift of a cector field on M to TM may be repeated with some modification T^* is a contra-variant functor (it reverses arrows). So we have to use the inverse map when lifting the flow.

$$T^*\left(\Phi_t^{-1}\right) \equiv T^*\Phi_{-t} \tag{13}$$

$$\begin{tikzcd} T^*M \arrow[r, "T^*\Phi_t"] \arrow[d, "\tau"'] & T^*M \arrow[d, "\tau"'] \\ M \arrow[r, "\Phi_t"] & M \end{tikzcd}$$

It can also be shown by similar arguments as before that $T^*\Phi_{-t}$ is indeed a flow on T^*M. We can also derive its vector field.

Consider the flow $\Phi_t : M \to M$. The infinitesimal flow reads

$$T^*\Phi_{-\epsilon} : (x^a, p_a) \to \left(x^a(\epsilon), p_a(\epsilon) = x^a + \epsilon V^a, p^a - \epsilon V^a_{,b} p^b\right)$$

so $V = V^a\partial_a$ the generator of the flow Φ_t is lifted to $V = V^a\partial_a - V^b_{,a} p_a \frac{\partial}{\partial p^a}$

4 Canonical Tensor Fields on TM and T^*M

We introduce the following flows

$$\Phi_t : v \longmapsto e^t v \quad p \longmapsto e^t p \tag{14}$$

The generators for the motion are $\triangle \in \Xi(TM)$ and $\triangle_p \in \Xi(T^*M)$ which are

$$\triangle = v^a \frac{\partial}{\partial v^a}$$
$$\triangle_p = p_a \frac{\partial}{\partial p_a}$$

The fields are in both cases vertical. Consider a tensor field A on TM or (T^*M) which is homogeneous of degree k in fiber co-ordinates. This means that if we express the field in terms canonical co-ordinates and then substitute $(x^a, v^a) \longmapsto (x^a, \lambda v^a)$ or $(x^a, p_a) \longmapsto (x^a, \lambda p_a)$ we get $A \longmapsto \lambda^k A$. Examples are the following.

$$1.\tilde{V} = V^a \frac{\partial}{\partial x^a} + V^a_{,n} v^b \frac{\partial}{\partial v^a} \longmapsto V^a \frac{\partial}{\partial x^a} + V^a_{,b} \lambda v^b \frac{\partial}{\partial \lambda v^a} \implies \tilde{V} \to \tilde{V}, k = 1$$
$$2.\alpha^{\uparrow} = \alpha_a(x) \frac{\partial}{\partial p_a} \longmapsto \alpha_a(x) \frac{\partial}{\lambda p_a} \implies \alpha^{\uparrow} \to \lambda^{-1} \alpha^{\uparrow}, k = -1$$
$$3.\hat{g} = g_{ab}(x) v^a v^b \longmapsto g_{ab}(x) \lambda^2 v^a v^b \implies \hat{g} \to \lambda^2 \hat{g}, k = 2$$

This means that $\mathcal{L}_{\triangle} A = kA$ Why? For $\Psi : (x^a, v^a) \longmapsto (x^a, \lambda v^a)$ or $\Psi : (x^a, p^a) \longmapsto (x^a, \lambda p^a)$. This corresponds to a flow with following effect $\Phi^* A = \lambda^k A$, set $\lambda^k = e^{kt}$. The lie derivative is the derivative with respect to time of the pullback of tensor on some flow evaluated at $t = 0$. So

$$\frac{d\Phi * A}{dt}|_{t=0} = ke^{kt} A$$

which implies the result.

The next canonical field on TM is a $\binom{1}{1}$ type tensor field which is called a vertical endomorphism $S \in \mathcal{T}^1_1(TM)$ defined as

$$S := \mathbb{I}^{\uparrow} \tag{15}$$

In canonical co-ordinates it is $S = dx^a \otimes \frac{\partial}{\partial v^a}$. We have the following results easily derived

$$S\left(\frac{\partial}{\partial x^a}\right) = \frac{\partial}{\partial v^a}$$
$$S\left(\frac{\partial}{\partial v^a}\right) = 0$$
$$S\left(dx^a\right) = 0$$
$$S\left(dv^a\right) = dx^a$$

Notice therefore that $KerS_v = ImS_v = VerTM$. Also S_v is nilpotent i.e $S_v \circ S_v = 0$ also note that $S(\tilde{V}) = V^{\uparrow}$. It is homogeneous and of degree -1.

5 Hamilton and Euler Lagrange Equations on Fiber Bundles

Consider a system of second-order differential equations

$$\ddot{x}^a = \Gamma^a(x, \dot{x}) \qquad a = 1, \ldots n \tag{16}$$

We introduce a new variable $v^a := \dot{x}^a(t)$. Then these system becomes a system of 2n-first order differential equations

$$\begin{aligned} \dot{x}^a &= v^a \\ \dot{v}^a &= \Gamma^a(x, v) \end{aligned}$$

We form a vector field with these equations

$$\Gamma = v^a \frac{\partial}{\partial x^a} + \Gamma^a(x, v) \frac{\partial}{\partial v^a} \tag{17}$$

If x^a is treated as a co-ordinate on the manifold then v^a is a natural co-ordinate on the tangent bundle. **Note:** Not all vector fields on TM are the kind we want.

If $W = A^a(x, v)\frac{\partial}{\partial x^a} + B^a(x, v)\frac{\partial}{\partial v^a}$ is a general vector field on TM then looking at Γ we see that the constraint is in $A^(a)(x, v)$, namely it should be $A^a(x, v) = v^a = \dot{x}^a$. This constraint can be summarized as follows $S(\Gamma) = \triangle$ where $\triangle$ is the Liouville field introduced earlier. If $\Gamma \in \Xi(TM)$ satsfies $S(\Gamma) = \triangle$ then each integral curve of the field Γ is the natural lift $\hat{\gamma}$ of a curve γ on M. Since $\Gamma = \dot{x}^a \frac{\partial}{\partial x^a} + \dot{v}^a \frac{\partial}{\partial v^a}$, then $\gamma(t) = (x^1(t), \ldots, x^n(t)$ on M, then we see that the integral curve of Γ is $\dot{x}^1(t), \ldots x^n(t)$ which is a map $\mathbb{R} \to TM$ but also the natural lift of $\gamma(t)$. All this means that second order differential equations are in 1-1 correspondence with vector fields on TM which satisfy $S(\Gamma) = \triangle$.

Example:

$$\begin{aligned} \ddot{x}^a = -\omega^2 x \implies \Gamma &= \dot{x}\frac{\partial}{\partial x} - \omega^2 x \frac{\partial}{\partial \dot{x}} \\ \Gamma &= (A\cos\omega t - B\sin\omega t)\frac{\partial}{\partial x} + (B\cos\omega t + A\sin\omega t)\frac{\partial}{\partial \dot{x}} \end{aligned}$$

5.1 Euler Lagrange Field

We have the equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^a} - \frac{\partial L}{\partial x^a} = 0 \tag{18}$$

If $L(x, \dot{x})$ is a Lagrangian and if

$$\begin{aligned}
A_{ab}(x, \dot{x}) &:= \frac{\partial^2 L}{\partial \dot{x}^a \partial \dot{x}^b} \\
A_{ab}(x, \dot{x}) &:= \frac{\partial^2 L}{\partial \dot{x}^a \partial x^b} \\
C_a(x, \dot{x}) &:= \frac{\partial L}{\partial x^a}
\end{aligned}$$

We can construct what the second order differential equation looks like, thusly,

$$\begin{aligned}
\frac{d}{dt} &= \frac{dt}{dt}\frac{\partial}{\partial t} + \frac{\partial x^a}{\partial t}\frac{\partial}{\partial x^a} + \frac{\partial \dot{x}^a}{\partial t}\frac{\partial}{\partial \dot{x}^a} \\
\left(\frac{dt}{dt}\frac{\partial}{\partial t} + \frac{\partial x^b}{\partial t}\frac{\partial}{\partial x^b} + \frac{\partial \dot{x}^b}{\partial t}\frac{\partial}{\partial \dot{x}^b}\right)\frac{\partial L}{\partial \dot{x}^a} - \frac{\partial L}{\partial x^a} &= 0 \\
\frac{\partial^2 L}{\partial x^b \partial \dot{x}^b}\dot{x}^b + \ddot{x}^b \frac{\partial^2 L}{\partial \dot{x}^b \partial \dot{x}^a} - \frac{\partial L}{\partial x^a} &= 0 \\
\ddot{x}^b \frac{\partial^2 L}{\partial \dot{x}^b \partial \dot{x}^a} &= -\frac{\partial^2 L}{\partial x^b \partial \dot{x}^a} + \frac{\partial L}{\partial x^a}
\end{aligned}$$

Playing around with indices eventually gices

$$\ddot{x}^a = \Gamma^a(x, \dot{x}) \qquad \text{where } \Gamma = -(A^{-1})^{ab} B_{ab}\dot{x}^c + (A^{-1})^{ab} C_b$$

Now we can show that with L we can introduce a symplectic structure.

To that end we introduce the cartan forms

$$\Theta_L := S(dL) \tag{19}$$

$$\omega := d\Theta_L \tag{20}$$

Simple calculations give

$$\begin{aligned}
\Theta_L &= \frac{\partial L}{\partial v^a} dx^a \\
\omega_L &= \frac{\partial^2 L}{\partial x^b \partial v^a} dx^b \wedge dx^a + \frac{\partial^2 L}{\partial v^a \partial v^a} dv^v \wedge dx^a
\end{aligned}$$

ω_L is closed since if one calculates $d\omega_L$ one of the co-ordinates will be repeated giving a result of zero. Plus ω_L is non-degenerate $\iff det\left(\frac{\partial^2 L}{\partial v^a \partial v^b} \neq 0\right)$ since $\omega_L \wedge \omega_L \wedge \cdots \wedge \omega_L \neq 0 \iff \omega_L \neq 0$. The above exterior product will look like $det\left(\frac{\partial^2 L}{\partial v^a \partial v^b}\right) dx^1 \wedge \cdots \wedge dx^n \wedge dv^1 \wedge \ldots dv^n \neq 0 \iff det\left(\frac{\partial^2 L}{\partial v^a \partial v^b}\right) \neq 0$

A Lagrangian that obeys the above conditions is called non-singular and also is non-singular $\iff$ ω_L is symplectic. So any non-singular or regular lagrangian makes a symplectic manifold (TM, ω_L) . The lagrangian we use in physics are regular.

We can introduce a Hamilton field ζ_f by

$$i_{\zeta_f}\omega_L = -df \tag{21}$$

Therefore we can make a Hamiltonian system (TM, ω_L, H) with $f = H$ and integral curves of ζ_H being the dynamics.

$$H := E_L = \triangle L - L \tag{22}$$

To show that eq 5.21 and 5.22 have embedded in them the second order ODE for the Euler Lagrange Equations we do the following.

$$\Gamma = A^i \frac{\partial}{\partial x^i} + B^i \frac{\partial}{\partial v^i}$$
$$i_\Gamma \omega_L = -dE_L$$

$$i_\Gamma \omega_L = \frac{\partial^2 L}{\partial v^a \partial v^b}\left(A^i \delta^i_a dv^b - B^i \delta^b_i dx^a\right) + \frac{\partial^2 L}{\partial x^a \partial v^b}\left(A^i \delta^a_i dx^b - A^i \delta^b_i dx^a\right) = -\left((\frac{\partial}{\partial v^c}(v^d \frac{\partial L}{\partial v^d}) - \frac{\partial L}{\partial v^c}) dv^c \right)$$
$$- \left(\frac{\partial}{\partial x^c}(v^d \frac{\partial L}{\partial v^d}) - \frac{\partial L}{\partial x^c}) dx^c \right)$$

$$A^i \left(\frac{\partial^2 L}{\partial v^i \partial v^b} dv^b + \frac{\partial^2 L}{\partial x^i \partial v^b} dx^b - \frac{\partial^2 L}{\partial x^a \partial v^i} dx^a \right) - \frac{\partial^2 L}{\partial v^a \partial v^i} dx^a B^i$$
$$= -\left(v^d \frac{\partial^2 L}{\partial v^c \partial v^d} + (v^d \frac{\partial^2 L}{\partial x^c \partial v^d} - \frac{\partial L}{\partial x^c}) dx^c \right)$$

Therefore we have

$$A^i \frac{\partial^2 L}{\partial v^i \partial v^b} dv^b = v^d \frac{\partial^2 L}{\partial v^c \partial v^d}$$
$$A^i \left[\frac{\partial^2 L}{\partial x^i \partial v^b} dx^b - \frac{\partial^2 L}{\partial x^a \partial v^c} \right] + B^i \left[\frac{\partial^2 L}{\partial v^a \partial v^i} \right] = \left[-v^d \frac{\partial^2 L}{\partial x^c \partial v^d} + \frac{\partial L}{\partial x^c} \right] dx^c$$

The above equation imply that $A^i = v^i$ and let us remember that $B^i = \frac{dv}{dt} = \frac{d^2 x^i}{dt}$ we get

$$\left[v^i \frac{\partial^2 L}{\partial x^i \partial v^b} + \ddot{x}^i \frac{\partial^2 L}{\partial v^a \partial v^i} \right] dx^i = \frac{\partial L}{\partial x^c} dx^c$$

Re-arranging the above equation gives us the ODE we wanted.

5.1.1 Co-ordinate Free Version of Euler Lagrange Equations

Define the map

$$\mathcal{E}^L : (V, \Gamma) \to \Gamma(V^{\uparrow} L) - \tilde{V} L \tag{23}$$

We write the co-ordinate presentation of the above map

$$\begin{aligned}
\Gamma(V^a \frac{\partial L}{\partial v^a}) - \left[v^a \frac{\partial L}{\partial x^a} + v^a_{,b} v^b \frac{\partial L}{\partial v^a} \right] \\
v^c \frac{\partial}{\partial x^c}(v^a \frac{\partial L}{\partial v^a}) + \dot{v}^c \frac{\partial}{\partial v^a}(v^a \frac{\partial L}{\partial v^a}) - \left[v^a \frac{\partial L}{\partial x^a} + v^a_{,b} v^b \frac{\partial L}{\partial v^a} \right] \\
v^a \left[v^c \frac{\partial}{\partial x^c} \frac{\partial L}{\partial v^a} + \dot{v}^c \frac{\partial}{\partial v^c} \frac{\partial L}{\partial v^a} - \frac{\partial L}{\partial x^a} - \dot{v}^a \frac{\partial L}{\partial v^a} \right] + \dot{v}^c \frac{\partial L}{\partial v^c} \\
v^a \left[\Gamma(\partial^{\uparrow}_a L) - \frac{\partial}{\partial x^a} \right] = v^a \mathcal{E}^L_a(x, v) = v^a \mathcal{E}(\partial_a, \Gamma)
\end{aligned}$$

The point is that you can pull out the v^a and everything will be fine. So we have $\mathcal{E}(\partial_a, \Gamma) = \Gamma \frac{\partial L}{\partial v^a} - \frac{\partial L}{\partial x^a}$. We can evaluate this on an integral curve of the field Γ, recall that $\dot{x}^c = v^c$ so $\dot{v}^c \frac{\partial}{\partial v^c} \frac{\partial L}{\partial v^a} = \frac{d}{dt}(\frac{\partial L}{\partial v^a})$ and $v^c \frac{\partial}{\partial x^c} \frac{\partial L}{\partial v^a} = 0$ so $\mathcal{E}(\partial_a, \Gamma) = \frac{d}{dt} \frac{\partial L}{\partial \dot{x} a} - \frac{\partial L}{\partial x^a}$ For the Euler Lagrange field we get the Euler-Lagrange equations.